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Mixed correlation function and spectral curve for the 2-matrix model

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Abstract
We compute the mixed correlation function in a way which involves only the orthogonal polynomials with degrees close to n (in some sense like the Christoffel–Darboux theorem for non-mixed correlation functions). We also derive new representations for the differential systems satisfied by the bi-orthogonal polynomials, and we find new formulae for the spectral curve. In particular we prove the conjecture of M Bertola, claiming that the spectral curve is the same curve which appears in the loop equations.

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1. Introduction

Consider a pair of random Hermitian matrices M_1 and M_2, of size n, with the probability measure

$$\exp(-\mathrm{Tr}(V_1(M_1) + V_2(M_2) + M_1 M_2))\,\mathrm{d}M_1\,\mathrm{d}M_2. \tag{1.1}$$

This random 2-matrix model has many applications to physics (in particular in quantum gravity, i.e. statistical physics on a random surface and conformal field theory [15, 30]) and mathematics (bi-orthogonal polynomials [34]). Another important application of the 2-Hermitian matrix model comes from the fact that it is the analytical continuation of the complex-matrix model, which describes the Dyson gas at $\beta = 2$ and is an important model of Laplacian growth [38]. And the complex-matrix model plays a crucial role in the AdS/CFT correspondence, in the so-called BMN limit [31]. In that model, it is important to know how to compute mixed expectation values.

For all applications, one would like to be able to compute various expectation values. Some expectation values can be written in terms of eigenvalues of M_1 and M_2, for instance $\langle \mathrm{Tr}\, M_1^k \mathrm{Tr}\, M_2^l \rangle$, which we call non-mixed because each trace contains only one type of matrix M_1 or M_2 but not both. In contrast, mixed expectation values are those where M_1 and M_2 may appear together in the same trace, for instance $\langle \mathrm{Tr}\, M_1^k M_2^l \rangle$. Mixed expectation values cannot be written in terms of eigenvalues of M_1 and M_2, and are thus more difficult to compute than non-mixed ones.

Beyond the technical challenge of computing them, mixed expectation values should play an important role in applications to boundary conformal field theory and to the BMN limit of string theory/CFT correspondence. While many formulae for non-mixed expectation values have been known for a long time, in particular in terms of bi-orthogonal polynomials [1–4, 16, 23, 37], it is only recently that formulae have been found for mixed traces. In particular, the following expectation value

$$W_n(x, y) = 1 + \left\langle \mathrm{Tr}\, \frac{1}{x - M_1} \frac{1}{y - M_2} \right\rangle \tag{1.2}$$

was first computed in [8]. The idea was to diagonalize the Hermitian matrices $M_1 = V X V^\dagger$ and $M_2 = V U Y U^\dagger V^\dagger$ where U and V are unitary matrices and X and Y are diagonal matrices containing the eigenvalues of M_1 and M_2. Then,

$$W_n(x, y) = 1 + \left\langle \mathrm{Tr}\, \frac{1}{x - X} U \frac{1}{y - Y} U^\dagger \right\rangle, \tag{1.3}$$

and using the Morozov's formula [19, 35] for unitary integrals of the form $\langle U_{ij} U^\dagger_{kl} \rangle$ one reexpresses W_n in terms of eigenvalues of M_1 and M_2 only. Then, the integration over eigenvalues is done with the method of bi-orthogonal polynomials [33]. The result found in [8] is thus an $n \times n$ determinant involving recursion coefficients of bi-orthogonal polynomials:

$$W_n(x, y) = \det_{n\times n} \left(\mathrm{Id}_n + \Pi_{n-1} \frac{1}{x - Q} \frac{1}{y - P^t} \Pi_{n-1} \right), \tag{1.4}$$

where the matrices Q and P implement the recursion relation (multiplication by x and y) of the bi-orthogonal polynomials, and Π_{n-1} is the projection on the polynomials of degree $\leqslant n - 1$. Those notation are explained in more details in section 2.4.

The mere existence of such a formula was a progress, but an $n \times n$ determinant is not convenient for practical computations and for taking large n limits. In the non-mixed case, the Christoffel–Darboux theorem allows one to rewrite expectation values in terms of smaller determinants, whose size does not grow with n [3, 4, 16].

The purpose of the present paper is to find a similar property for $W_n(x, y)$, i.e. write it in terms of determinants whose size is independent of n.

As a byproduct of such a rewriting, we are able to find new formulae for the spectral curve of the isomonodromic differential system satisfied by the corresponding bi-orthogonal polynomials.

1.1. Plan

- In section 2, we recall the definitions of bi-orthogonal polynomials, as well as their Fourier–Laplace transforms, Cauchy transforms and the third type of solution introduced by [13, 25]. We also recall their recursion relations, obtained by multiplication by x or derivation. We define the infinite matrices (finite band) Q and P, which encode the recursion relations (Q and P are also Lax matrices).
- In section 3, we study the inverse of $x - Q$ and $y - P$, in particular, we show that they have distinct right and left inverses. The difference between the right and left inverse is related to the so-called folding matrix [11] and is found to be a sum of bi-orthogonal polynomials and their various transforms.
- In section 4, we introduce the kernels [16], which are the building blocks of correlation functions. We show that they satisfy some Christoffel–Darboux relations.

- In section 5, we compute the bi-orthogonal polynomials, their Cauchy transforms and the kernels as determinants involving matrices Q and P.
- In section 6, we introduce the notion of windows of consecutive bi-orthogonal polynomials [9–11], because the recursion relations (matrix Q and P) allow us to rewrite any bi-orthogonal polynomial p_m as a linear combination of p_j with $n - d_2 \leqslant j \leqslant n$, where $d_2 + 1$ is the degree of the potential entering the weight of orthogonality.
- In section 7, we prove our main result. We compute the mixed correlation function $W_n(x, y)$ in terms of polynomials in the window and in terms of the kernels.
- In section 8, we study some consequences of that formula. In particular, we find new representations of the differential system satisfied by a window and we compute the spectral curve. We find several new formulae for the spectral curve and we prove the conjecture of M Bertola, which claims that the spectral curve is the same curve which was found from loop equations.
- In section 9, we discuss the consequences of that formula in terms of tau functions.
- In section 10, we give some explicit examples of our formulae, namely the Gaussian and Gaussian elliptical cases.
- Most of the technical proofs are put in the appendix.

1.2. Main results

This paper is very technical, and the purpose is to give many formulae for effective computations with bi-orthogonal polynomials. We propose some new formulae in almost every paragraph. Let us mention here the most important ones:

- In section 3.1, we give integral representations of the right (R) and left (L) inverses of $x - Q$. One of the important relations is equation (3.30):

$$L(x) - R(x) = \Psi_\infty(x)\hat{\Phi}^t_\infty(x). \tag{1.5}$$

- In section 5, we give determinantal expressions of bi-orthogonal polynomials and their transforms, as well as kernels. For instance the kernel $K_n(x, y)$ is

$$K_{n+1}(x, y) \propto \det_{n\times n}((x - Q)(y - P^t)). \tag{1.6}$$

- In section 7, we state one of the main results of this paper, i.e. some formulae for the two-point mixed correlation function $W_n(x, y)$:

$$\begin{aligned} W_n(x, y) &= 1 + \left\langle \mathrm{Tr}\, \frac{1}{x - M_1}\frac{1}{y - M_2} \right\rangle \\ &= \det_{n\times n}\left(1 + \frac{1}{x - Q}\frac{1}{y - P^t}\right) \\ &= \gamma_n^2 K_{n+1}(x, y) J_{n-1}(x, y) \\ &\quad\times \det_{d_2+1\times d_2+1}\left(1 - \Pi_n^{n-d_2}\left(1 - \frac{\vec{\psi}_n\vec{\phi}^t_n}{K_{n+1}}\right)U_n\left(1 - \frac{\hat{\vec{\psi}}_n\hat{\vec{\phi}}^t_n}{J_{n-1}}\right)\tilde{U}^t_n\right) \end{aligned} \tag{1.7}$$

  and theorem 7.2 gives W_n in terms of kernels only.
  We also find interesting recursion relations for $W_{n+1} - W_n$.
- In section 8, we compute new representations of the differential system $\mathcal{D}_n(x) = \Psi'_n(x)\Psi_n(x)^{-1}$ and we compute the spectral curve:

$$\mathcal{E}_n(x, y) = \tilde{t}\,\det(y - \mathcal{D}_n(x)) = -\gamma_n^2 \det\left(1 - U_n\tilde{U}^t_n\right) = \det(y - H^{(ij)}(x, x)). \tag{1.8}$$

The most interesting is that we prove Bertola's conjecture [5]:

$$\mathcal{E}_n(x,y) = (V_1'(x)+y)(V_2'(y)+x) - n + \left\langle \mathrm{Tr}\, \frac{V_1'(x)-V_1'(M_1)}{x-M_1}\, \frac{V_2'(y)-V_2'(M_2)}{y-M_2} \right\rangle. \quad (1.9)$$

This conjecture has important consequences in terms of tau functions. Indeed, the Miwa–Jimbo–Ueno approach of isomonodromic tau functions [27, 28, 36], generalized in [7], allows us to express the tau function in terms of residues of the spectral curve, and this formula is particularly convenient for that purpose. It shows that the tau function is the matrix integral and it shows that some additional parameters could be added to the model.

2. Definitions and notation about bi-orthogonal polynomials

This section recalls well-known facts about bi-orthogonal polynomials and stands here just for setting notation and describing known properties. Notation are similar (although with small differences) to those of [10, 11].

2.1. Measure and integration paths

Consider the weight

$$\omega(x,y) = \exp(-(V_1(x)+V_2(y)+xy)), \quad (2.1)$$

where V_1 is a complex polynomial of degree d_1+1 and V_2 is a complex polynomial of degree d_2+1:

$$V_1(x) = \sum_{k=0}^{d_1+1} t_k x^k, \qquad V_2(y) = \sum_{k=0}^{d_2+1} \tilde{t}_k y^k. \quad (2.2)$$

We write the leading coefficients of V_1' and V_2':

$$t = (d_1+1)t_{d_1+1}, \qquad \tilde{t} = (d_2+1)\tilde{t}_{d_2+1}. \quad (2.3)$$

We choose a basis of d_1 contours $\gamma^{(i)}$ with $i = 1,\dots,d_1$, going from ∞ to ∞ in sectors where the integral $\int \mathrm{e}^{-V_1(x)}\,\mathrm{d}x$ is convergent, and we choose a basis of d_2 contours $\tilde{\gamma}^{(i)}$ with $i = 1,\dots,d_2$, going from ∞ to ∞ in sectors where the integral $\int \mathrm{e}^{-V_2(y)}\,\mathrm{d}y$ is convergent (see [6, 10]).

Then we choose a dual basis of d_1 contours $\overline{\gamma}^{(i)}$ with $i = 1,\dots,d_1$, going from ∞ to ∞ in sectors where the integral $\int \mathrm{e}^{+V_1(x)}\,\mathrm{d}x$ is convergent, and we choose a dual basis of d_2 contours $\overline{\tilde{\gamma}}^{(i)}$ with $i = 1,\dots,d_2$, going from ∞ to ∞ in sectors where the integral $\int \mathrm{e}^{+V_2(y)}\,\mathrm{d}y$ is convergent, such that

$$\gamma^{(i)} \cap \overline{\gamma}^{(j)} = \delta_{ij}, \qquad \tilde{\gamma}^{(i)} \cap \overline{\tilde{\gamma}}^{(j)} = \delta_{ij}. \quad (2.4)$$

Then, we choose $d_1 d_2$ numbers $\kappa_{i,j}$ such that at least one of them is non-vanishing, and we define a path Γ:

$$\Gamma := \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} \kappa_{i,j}\gamma^{(i)} \times \tilde{\gamma}^{(j)}, \quad (2.5)$$

and we define the following measure on Γ:

$$\mathrm{d}\mu(x,y) = \exp(-(V_1(x)+V_2(y)+xy))\,\mathrm{d}x\,\mathrm{d}y. \quad (2.6)$$

Remark 2.1. Generalized path Γ and matrix models. We have introduced the generalized integration contours Γ, because it is the most general contour on which the measure $d\mu$ can be integrated. It corresponds to a generalization of the Hermitian 2-matrix model. Indeed, Hermitian matrices have their eigenvalues on the real axis, and the Hermitian 2-matrix model equation (1.1) corresponds to the case $\Gamma = \mathbf{R} \times \mathbf{R}$.

A generalized path Γ can also correspond to a matrix model, with matrices which are not Hermitian. It corresponds to normal matrices (i.e. which can be diagonalized by a unitary transformation, but with complex eigenvalues), with pairs of eigenvalues constrained to be on Γ. This allows us to define an ensemble of matrices which is noted $H_n \times H_n(\Gamma)$, see [18] for more details.

In this normal matrix model, it makes sense to compute matrix expectation values, in particular the mixed correlation function $W_n(x, y) = 1 + \left\langle \mathrm{Tr}\, \frac{1}{x-M_1} \frac{1}{y-M_2} \right\rangle$.

2.1.1. More definitions. We define (see [13, 25])

$$h_0 = \int_\Gamma \mathrm{d}\mu(x, y) = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \kappa_{i,j} \int_{\gamma^{(i)}} \mathrm{d}x \int_{\tilde\gamma^{(j)}} \mathrm{d}y \exp(-(V_1(x) + V_2(y) + xy)). \tag{2.7}$$

And

$$g^{(i)}(x) := \frac{\sqrt{h_0}}{2\mathrm{i}\pi} \int_{\bar{\tilde\gamma}^{(i)}} \mathrm{e}^{xy}\, \mathrm{e}^{V_2(y)}\, \mathrm{d}y, \qquad i = 1, \ldots, d_2, \qquad g^{(0)} := 0, \tag{2.8}$$

$$\tilde g^{(i)}(y) := \frac{\sqrt{h_0}}{2\mathrm{i}\pi} \int_{\bar\gamma^{(i)}} \mathrm{e}^{xy}\, \mathrm{e}^{V_1(x)}\, \mathrm{d}x, \qquad i = 1, \ldots, d_1, \qquad \tilde g^{(0)} := 0, \tag{2.9}$$

which are the independent solutions of the differential equations:

$$V_2'(\partial/\partial x) g^{(i)}(x) = -x g^{(i)}(x) \tag{2.10}$$

$$V_1'(\partial/\partial y) \tilde g^{(i)}(y) = -y \tilde g^{(i)}(y). \tag{2.11}$$

Then we define the following 'concomitants' [13, 25]:

$$c^{(ij)} = \frac{1}{2\mathrm{i}\pi} \int_{\gamma^{(i)}} \mathrm{d}x \int_{\bar\gamma^{(j)}} \mathrm{d}x' \frac{V_1'(x) - V_1'(x')}{x - x'}\, \mathrm{e}^{V_1(x') - V_1(x)}\, \mathrm{e}^{(x'-x)y}, \tag{2.12}$$

$$\tilde c^{(ij)} = \frac{1}{2\mathrm{i}\pi} \int_{\tilde\gamma^{(i)}} \mathrm{d}y \int_{\bar{\tilde\gamma}^{(j)}} \mathrm{d}y' \frac{V_2'(y) - V_2'(y')}{y - y'}\, \mathrm{e}^{V_2(y') - V_2(y)}\, \mathrm{e}^{(y'-y)x}. \tag{2.13}$$

$c^{(ij)}$'s (resp. $\tilde c^{(ij)}$'s) are independent of y (resp. x). Due to the dual choice of contours equation (2.4), they are normalized:

$$c^{(ij)} = \delta_{i,j}, \qquad \tilde c^{(ij)} = \delta_{i,j}. \tag{2.14}$$

Indeed, integrating by parts, we can replace both $V_1'(x)$ by y and $V_1'(x')$ by y. If $i \neq j$, the contours $\gamma^{(i)}$ and $\bar\gamma^{(j)}$ do not intersect, and the integration by parts gives no boundary term and the result vanishes. If $i = j$, the two contours intersect and we have a boundary term. A way to compute it is to write the pole $1/(x - x')$ as the sum of a principal part and $2\mathrm{i}\pi\delta(x - x')$. The principal part is integrated by parts and gives zero as in the $i \neq j$ case, whilst the δ-term corresponds to the boundary term in the integration by parts and it gives 1.

This computation was first done by the authors of [13, 25], in the comparison of the two Riemann–Hilbert problems derived for bi-orthogonal polynomials [10, 29, 32].

2.2. Bi-orthogonal polynomials

The monic bi-orthogonal polynomials [33, 34] (if they exist) are uniquely determined by

$$p_n(x) = x^n + \cdots, \qquad q_n(y) = y^n + \cdots, \tag{2.15}$$

and

$$\int_\Gamma p_n(x) q_m(y)\, \mathrm{d}\mu(x, y) = h_n \delta_{nm}. \tag{2.16}$$

For given potentials V_1 and V_2, bi-orthogonal polynomials exist for almost every choice of Γ (in fact they do not exist only for an enumerable set of Γ's, see [10]).

2.2.1. Wavefunctions. We define

$$\psi_n(x) := \frac{1}{\sqrt{h_n}} p_n(x)\, \mathrm{e}^{-V_1(x)}, \qquad \phi_n(y) := \frac{1}{\sqrt{h_n}} q_n(y)\, \mathrm{e}^{-V_2(y)}. \tag{2.17}$$

2.2.2. Cauchy transforms. We introduce the Cauchy transforms [9, 10]:

$$\hat{\psi}_n(y) := \frac{1}{\sqrt{h_n}}\, \mathrm{e}^{V_2(y)} \int_\Gamma \frac{1}{y - y'} p_n(x')\, \mathrm{d}\mu(x', y'), \tag{2.18}$$

$$\hat{\phi}_n(x) := \frac{1}{\sqrt{h_n}}\, \mathrm{e}^{V_1(x)} \int_\Gamma \frac{1}{x - x'} q_n(y')\, \mathrm{d}\mu(x', y'). \tag{2.19}$$

2.2.3. Fourier–Laplace transforms. We also introduce the following functions [16, 17]:

$$\hat{\psi}_n^{(i)}(y) := \int_{\gamma^{(i)}} \psi_n(x)\, \mathrm{e}^{-xy}\, \mathrm{d}x \qquad \text{for} \quad i = 1, \ldots, d_1, \tag{2.20}$$

$$\hat{\phi}_n^{(i)}(x) := \int_{\tilde{\gamma}^{(i)}} \phi_n(y)\, \mathrm{e}^{-xy}\, \mathrm{d}y \qquad \text{for} \quad i = 1, \ldots, d_2. \tag{2.21}$$

They are the Fourier–Laplace transforms of ψ_n and ϕ_n. For $i = 0$ we will also write

$$\hat{\psi}_n^{(0)}(y) := \hat{\psi}_n(y), \qquad \hat{\phi}_n^{(0)}(x) := \hat{\phi}_n(x). \tag{2.22}$$

2.2.4. Third-type functions. The authors of [13, 25] have introduced the following functions:

$$\phi_n^{(i)}(y) := \frac{1}{2\mathrm{i}\pi} \frac{1}{\sqrt{h_n}} \int_{\overline{\gamma}^{(i)}} \mathrm{d}x \int_\Gamma \mathrm{d}\mu(x', y') \frac{1}{y - y'} \frac{V_1'(x) - V_1'(x')}{x - x'}\, \mathrm{e}^{V_1(x)}\, \mathrm{e}^{xy} q_n(y'), \tag{2.23}$$

$$\psi_n^{(i)}(x) := \frac{1}{2\mathrm{i}\pi} \frac{1}{\sqrt{h_n}} \int_{\overline{\tilde{\gamma}}^{(i)}} \mathrm{d}y \int_\Gamma \mathrm{d}\mu(x', y') \frac{1}{x - x'} \frac{V_2'(y) - V_2'(y')}{y - y'}\, \mathrm{e}^{V_2(y)}\, \mathrm{e}^{xy} p_n(x'). \tag{2.24}$$

We will also write

$$\psi_n^{(0)}(x) := \psi_n(x), \qquad \phi_n^{(0)}(y) := \phi_n(y). \tag{2.25}$$

Formally, those functions are the 'inverse Fourier transforms' of $\hat{\psi}$'s, as described in [10, 29].

2.3. Semi-infinite vectors and matrices

We introduce semi-infinite vector notation:

$$\psi_\infty(x) = (\psi_0(x), \psi_1(x), \psi_2(x), \ldots)^t, \qquad \phi_\infty(y) = (\phi_0(y), \phi_1(y), \phi_2(y), \ldots)^t, \tag{2.26}$$

and more generally

$$\begin{aligned} \psi_\infty^{(i)}(x) &= \left(\psi_0^{(i)}(x), \psi_1^{(i)}(x), \psi_2^{(i)}(x), \ldots\right)^t, \\ \phi_\infty^{(i)}(y) &= \left(\phi_0^{(i)}(y), \phi_1^{(i)}(y), \phi_2^{(i)}(y), \ldots\right)^t. \end{aligned} \tag{2.27}$$

And

$$\hat\psi_\infty(y) = (\hat\psi_0(y), \hat\psi_1(y), \hat\psi_2(y), \ldots)^t, \qquad \hat\phi_\infty(x) = (\hat\phi_0(x), \hat\phi_1(x), \hat\phi_2(x), \ldots)^t, \tag{2.28}$$

and more generally

$$\begin{aligned} \hat\psi_\infty^{(i)}(y) &= \left(\hat\psi_0^{(i)}(y), \hat\psi_1^{(i)}(y), \hat\psi_2^{(i)}(y), \ldots\right)^t, \\ \hat\phi_\infty^{(i)}(x) &= \left(\hat\phi_0^{(i)}(x), \hat\phi_1^{(i)}(x), \hat\phi_2^{(i)}(x), \ldots\right)^t. \end{aligned} \tag{2.29}$$

We also introduce the basis vectors:

$$\mathbf{e}_n = (\overbrace{0, \ldots, 0}^{n}, 1, \overbrace{0, \ldots}^{\infty})^t, \tag{2.30}$$

i.e. the vector whose only non-vanishing component is in the $n + 1$ position. It is such that

$$\mathbf{e}_n^t \psi_\infty(x) = \psi_n(x). \tag{2.31}$$

Similarly, we consider the projection matrix:

$$\Pi_n = \mathrm{diag}(\overbrace{1, \ldots, 1}^{n+1}, \overbrace{0, \ldots}^{\infty}) = \sum_{j=0}^{n} \mathbf{e}_j\, \mathbf{e}_j^t, \tag{2.32}$$

with $n + 1$ ones on the diagonal. It is the projector on the span of $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$. We also define

$$\Pi^n = 1 - \Pi_{n-1} = \text{projector on the span of } n, n+1, \ldots, \infty, \tag{2.33}$$

and

$$\Pi_m^n = \sum_{j=n}^{m} \mathbf{e}_j\, \mathbf{e}_j^t = \Pi^n \Pi_m = \Pi_m - \Pi_{n-1} = \text{projector on the span of } n, \ldots, m. \tag{2.34}$$

We also introduce the following $\infty \times (d_2 + 1)$ (resp. $\infty \times (d_1 + 1)$) matrices:

$$\Psi_\infty := \left(\psi_\infty^{(0)} \quad \psi_\infty^{(1)} \quad \cdots \quad \psi_\infty^{(d_2)}\right), \tag{2.35}$$

$$\Phi_\infty := \left(\phi_\infty^{(0)} \quad \phi_\infty^{(1)} \quad \cdots \quad \phi_\infty^{(d_1)}\right), \tag{2.36}$$

$$\hat\Psi_\infty := \left(\hat\psi_\infty^{(0)} \quad \hat\psi_\infty^{(1)} \quad \cdots \quad \hat\psi_\infty^{(d_1)}\right), \tag{2.37}$$

$$\hat\Phi_\infty := \left(\hat\phi_\infty^{(0)} \quad \hat\phi_\infty^{(1)} \quad \cdots \quad \hat\phi_\infty^{(d_2)}\right). \tag{2.38}$$

2.4. Recursion relations for the bi-orthogonal polynomials

It is well known[1] that we have the following recursion relations [9, 15]:

$$x\psi_n = \sum_m Q_{nm}\psi_m, \qquad y\phi_n = \sum_m P_{nm}\phi_m, \tag{2.39}$$

$$\psi_n' = \sum_m P_{mn}\psi_m, \qquad \phi_n' = \sum_m Q_{mn}\phi_m. \tag{2.40}$$

Since V_1 and V_2 are polynomials, Q and P must be finite-band matrices, i.e.,

$$Q_{nm} \neq 0 \qquad \text{iff} \quad n - d_2 \leqslant m \leqslant n+1, \tag{2.41}$$

$$P_{nm} \neq 0 \qquad \text{iff} \quad n - d_1 \leqslant m \leqslant n+1. \tag{2.42}$$

In vector notation we have

$$x\psi_\infty = Q\psi_\infty, \qquad y\phi_\infty = P\phi_\infty, \tag{2.43}$$

$$\psi_\infty' = P^t\psi_\infty, \qquad \phi_\infty' = Q^t\phi_\infty. \tag{2.44}$$

2.5. Relations between Q and P

The matrices Q and P have the following properties [9, 10, 15]:

$$[Q, P^t] = \text{Id}. \tag{2.45}$$

$$Q_{n,n+1} = P_{n,n+1} = \sqrt{\frac{h_{n+1}}{h_n}} := \gamma_{n+1}. \tag{2.46}$$

$P^t + V_1'(Q)$ is a strictly lower triangular matrix

$$(P^t + V_1'(Q))_+ = 0, \qquad (P^t + V_1'(Q))_{n,n-1} = \frac{n}{\gamma_n}, \tag{2.47}$$

and $Q^t + V_2'(P)$ is a strictly lower triangular matrix

$$(Q^t + V_2'(P))_+ = 0, \qquad (Q^t + V_2'(P))_{n,n-1} = \frac{n}{\gamma_n}. \tag{2.48}$$

2.6. Recursion relations for the Cauchy transforms

The Cauchy transforms also satisfy recursion relations [10]:

$$y\hat{\psi}_\infty(y) = P^t\hat{\psi}_\infty(y) + \frac{1}{\phi_0(y)}\,\mathbf{e}_0, \qquad x\hat{\phi}_\infty(x) = Q^t\hat{\phi}_\infty(x) + \frac{1}{\psi_0(x)}\,\mathbf{e}_0, \tag{2.49}$$

$$\hat{\psi}_\infty' = -Q\hat{\psi}_\infty + \frac{1}{\phi_0}\frac{V_2'(y) - V_2'(P^t)}{y - P^t}\,\mathbf{e}_0, \tag{2.50}$$

$$\hat{\phi}_\infty' = -P\hat{\phi}_\infty + \frac{1}{\psi_0}\frac{V_1'(x) - V_1'(Q^t)}{x - Q^t}\,\mathbf{e}_0. \tag{2.51}$$

[1] x and ∂_x acting on a polynomial give a polynomial which can be decomposed on the basis of bi-orthogonal polynomials.

2.7. Recursion relations for the Fourier–Laplace transforms

For $i \neq 0$ we have [9, 10]

$$y\hat{\psi}_\infty^{(i)} = P^t \hat{\psi}_\infty^{(i)}, \qquad x\hat{\phi}_\infty^{(i)} = Q^t \hat{\phi}_\infty^{(i)}, \tag{2.52}$$

$$\hat{\psi}_\infty^{\prime(i)} = -Q\hat{\psi}_\infty^{(i)}, \qquad \hat{\phi}_\infty^{\prime(i)} = -P\hat{\phi}_\infty^{(i)}. \tag{2.53}$$

Note that they satisfy the same recursion relation as the Cauchy transforms without the non-homogeneous term.

2.8. Recursion relations for the third-type functions

For $i \neq 0$ we have [10, 13, 25]

$$\begin{aligned} x\psi_\infty^{(i)} &= Q\psi_\infty^{(i)} + \left(\frac{V_2'(\partial/\partial x) - V_2'(P^t)}{\partial/\partial x - P^t}\right)\mathbf{e}_0 g^{(i)} \\ &= Q\psi_\infty^{(i)} + \frac{\sqrt{h_0}}{2\mathrm{i}\pi}\int_{\tilde{\gamma}^{(i)}} \mathrm{d}y\, \mathrm{e}^{V_2(y)}\, \mathrm{e}^{xy}\left(\frac{V_2'(y) - V_2'(P^t)}{y - P^t}\right)\mathbf{e}_0, \end{aligned} \tag{2.54}$$

$$\psi_\infty^{\prime(i)} = P^t\psi_\infty^{(i)} - g^{(i)}\,\mathbf{e}_0, \tag{2.55}$$

$$y\phi_\infty^{(i)} = P\phi_\infty^{(i)} + \left(\frac{V_1'(\partial/\partial y) - V_1'(Q^t)}{\partial/\partial y - Q^t}\right)\mathbf{e}_0\tilde{g}^{(i)}, \tag{2.56}$$

$$\phi_\infty^{\prime(i)} = Q^t\phi_\infty^{(i)} - \tilde{g}^{(i)}\,\mathbf{e}_0. \tag{2.57}$$

Note that they satisfy the same recursion relation as the wavefunctions, with an additional non-homogeneous term.

3. Inverses

The formula for mixed correlation functions found in [8] is written in terms of the inverse operators of $x - Q$ and $y - P$, thus we study them in detail in this section. $x - Q$ and $y - P$ also have distinct right and left inverses, which were shown to play a crucial role in the notion of folding onto a window in [11].

3.1. Inverse

By definition, the infinite matrix $1/(x - Q)$ has elements

$$\left(\frac{1}{x - Q}\right)_{nm} = \frac{1}{\sqrt{h_n h_m}}\int_\Gamma q_m(y')\frac{1}{x - x'}p_n(x')\,\mathrm{d}\mu(x', y') = \psi_n(x)\hat{\phi}_m(x) + R_{nm}(x), \tag{3.1}$$

where

$$R_{nm}(x) := \frac{-1}{\sqrt{h_n h_m}}\int_\Gamma q_m(y')\frac{p_n(x) - p_n(x')}{x - x'}\,\mathrm{d}\mu(x', y') \tag{3.2}$$

which is a polynomial in x of degree $n - 1$.

In vector notation we have

$$\frac{1}{x - Q} = \psi_\infty(x)\hat{\phi}_\infty^t(x) + R(x). \tag{3.3}$$

Similarly,

$$\frac{1}{y-P} = \phi_\infty(y)\hat{\psi}^t_\infty(y) + \tilde{R}(y) \tag{3.4}$$

where

$$\tilde{R}_{mn}(y) := -\frac{1}{\sqrt{h_n h_m}} \int_\Gamma \frac{q_n(y)-q_n(y')}{y-y'} p_m(x')\,\mathrm{d}\mu(x',y'). \tag{3.5}$$

3.2. Right inverse

The semi-infinite matrix $R(x)$ (resp. $\tilde{R}(y)$) is polynomial in x (resp. y) and is strictly lower triangular:

$$R_{nm} = 0 \qquad \text{if} \quad m \geqslant n \qquad (\text{resp. } \tilde{R}_{nm} = 0 \quad \text{if} \quad m \geqslant n). \tag{3.6}$$

It is a right inverse [11] of $(x-Q)$ (resp. $(y-P)$):

$$(x-Q)R(x) = \mathrm{Id} \qquad (\text{resp. } (y-P)\tilde{R}(y) = \mathrm{Id}). \tag{3.7}$$

But it is not a left inverse, we have

$$R(x)(x-Q) = \mathrm{Id} - \frac{\psi_\infty(x)\,\mathbf{e}_0^t}{\psi_0(x)} \qquad \left(\text{resp. } \tilde{R}(y)(y-P) = \mathrm{Id} - \frac{\phi_\infty(y)\,\mathbf{e}_0^t}{\phi_0(y)}\right). \tag{3.8}$$

Note that

$$R_{n,n-1} = -\frac{1}{\gamma_n} = \tilde{R}_{n,n-1}. \tag{3.9}$$

Note that we have

$$\frac{\psi_n(x)-\psi_n(x')}{x-x'} = -\sum_m R_{nm}(x)\psi_m(x') \tag{3.10}$$

$$\left(\text{resp. } \frac{\phi_n(y)-\phi_n(y')}{y-y'} = -\sum_m \tilde{R}_{nm}(y)\phi_m(y')\right). \tag{3.11}$$

In particular, at $x = x'$

$$\psi'_\infty(x) = -R(x)\psi_\infty(x) = P^t\psi_\infty(x), \tag{3.12}$$

$$(\text{resp. } \phi'_\infty(y) = -\tilde{R}(y)\phi_\infty(y) = Q^t\phi_\infty(y)). \tag{3.13}$$

Also note that

$$R(x_1)R(x_2) = -\frac{R(x_1)-R(x_2)}{x_1-x_2} \tag{3.14}$$

and

$$[R(x_1), R(x_2)] = 0. \tag{3.15}$$

It can be found, by solving directly the system $(x-Q)R(x) = 1$ for a lower triangular matrix R, that

$$R_{nm}(x) = \begin{cases} \dfrac{-1}{\gamma_{m+1}\cdots\gamma_n}\det\left(\Pi_{m+1}^{n-1}(x-Q)\Pi_{m+1}^{n-1}\right) & \text{for} \quad m \leqslant n-2 \\ \dfrac{-1}{\gamma_n} & \text{for} \quad m = n-1 \\ 0 & \text{for} \quad m \geqslant n \end{cases}. \tag{3.16}$$

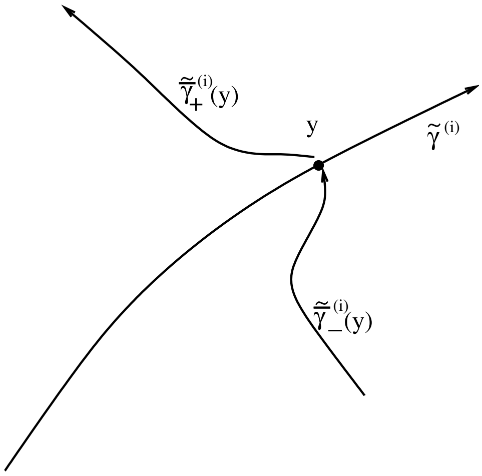

Figure 1. Definition of the contours $\tilde{\tilde{\gamma}}_+^{(i)}(y)$ and $\tilde{\tilde{\gamma}}_-^{(i)}(y)$.

3.3. Left inverse

Consider $y \in \tilde{\gamma}^{(i)}$, then deform the contour $\tilde{\tilde{\gamma}}^{(i)}$ so that it crosses $\tilde{\gamma}^{(i)}$ at y, and define $\tilde{\tilde{\gamma}}_-^{(i)}(y)$ as the part of $\tilde{\tilde{\gamma}}^{(i)}$ which stands on the right of $\tilde{\gamma}^{(i)}$ and which ends at y, and $\tilde{\tilde{\gamma}}_+^{(i)}(y)$ as the part of $\tilde{\tilde{\gamma}}^{(i)}$ which stands on the left of $\tilde{\gamma}^{(i)}$ and which starts at y (see figure 1). Then, define

$$L_{nm}(x) := \frac{-1}{2\mathrm{i}\pi}\sum_{i=1}^{d_2}\int_{\tilde{\gamma}^{(i)}} \mathrm{d}y\, \phi_m(y)\,\mathrm{e}^{-xy}\left(\int_{\tilde{\tilde{\gamma}}_+^{(j)}(y)}\mathrm{d}y'' + \int_{\tilde{\tilde{\gamma}}_-^{(j)}(y)}\mathrm{d}y''\right)\hat{\psi}_n(y'')\,\mathrm{e}^{xy''} \tag{3.17}$$

It has the following properties:

Theorem 3.1. *$L_{nm}(x)$ is a polynomial in x.*

L is upper triangular, with $L_{nm}(x) = 0$ if $n + d_2 > m$.

L is a left inverse of $x - Q$:

$$L(x)(x - Q) = 1. \tag{3.18}$$

We have

$$L_{nm}(x) = \left(\frac{1}{x - Q}\right)_{nm} + \sum_{i=1}^{d_2}\psi_n^{(i)}(x)\hat{\phi}_m^{(i)}(x). \tag{3.19}$$

Proof. The proof is given in appendix A. □

3.3.1. Alternative definition 1. It can be found, by solving directly the system $L(x)(x-Q) = 1$ for an upper triangular matrix L, that

$$L_{nm}(x) = \begin{cases} \dfrac{-1}{Q_{n+d_2,n}\cdots Q_{m,m-d_2}}\det\left(\Pi_{n+d_2}^{m-1}(x - Q)\Pi_{n+1}^{m-d_2}\right) & \text{for} \quad m > n + d_2 \\ \dfrac{-1}{Q_{n+d_2,n}} & \text{for} \quad m = n + d_2 \\ 0 & \text{for} \quad m < n + d_2 \end{cases}. \tag{3.20}$$

This proves that there is only one left inverse of $x - Q$ which is upper triangular.

3.3.2. Alternative definition 2. Define

$$l(x) := \sum_i \frac{\tilde{R}^t(y_i(x))}{V_2''(y_i(x))} \tag{3.21}$$

where $y_i(x)$ are the d_2 solutions of $V_2'(y) + x = 0$. Note that $l(x)$ is an upper triangular matrix such that

$$l_{nm}(x) = 0 \qquad \text{if} \quad n + d_2 > m. \tag{3.22}$$

The left inverse [11] of $x - Q$ is given by

$$\begin{aligned} L(x) &:= -l(x)\frac{1}{1 + (V_2'(P^t) + Q)l(x)} \\ &= -l(x) + l(x)(V_2'(P^t) + Q)l(x) \\ &\quad - l(x)(V_2'(P^t) + Q)l(x)(V_2'(P^t) + Q)l(x) + \cdots. \end{aligned} \tag{3.23}$$

Since $l(x)$ and $V_2'(P^t) + Q$ are strictly upper triangular matrices, each entry of that infinite sum is actually a finite sum. $L(x)$ is a strictly upper triangular matrix, such that

$$L_{nm}(x) = 0 \qquad \text{if} \quad n + d_2 > m. \tag{3.24}$$

This definition clearly gives a left inverse of $x - Q$, and since the upper triangular left inverse of $x - Q$ is unique, it must coincide with the first definition.

Remark. l computes the Euclidean division of q_m by $V_2'(y) + x$:

$$\phi_m(y) = -(V_2'(y) + x)\sum_n l_{nm}(x)\phi_n(y) + (\deg \leqslant d_2 - 1). \tag{3.25}$$

Similarly, we define

$$\tilde{l}(y) := \sum_i \frac{R^t(x_i(y))}{V_1''(x_i(y))}, \tag{3.26}$$

and the left inverse of $y - P$ is

$$\begin{aligned} \tilde{L}(y) &:= -\tilde{l}(y)\frac{1}{1 + (V_1'(Q^t) + P)\tilde{l}(y)} \\ &= -\tilde{l}(y) + \tilde{l}(y)(V_1'(Q^t) + P)\tilde{l}(y) \\ &\quad - \tilde{l}(y)(V_1'(Q^t) + P)\tilde{l}(y)(V_1'(Q^t) + P)\tilde{l}(y) + \cdots. \end{aligned} \tag{3.27}$$

3.4. Relationship between right and left inverses

We have

$$\left(\frac{1}{x - Q}\right)_{nm} = R_{nm}(x) + \psi_n^{(0)}(x)\hat{\phi}_m^{(0)}(x) = L_{nm}(x) - \sum_{i=1}^{d_2} \psi_n^{(i)}(x)\hat{\phi}_m^{(i)}(x), \tag{3.28}$$

i.e. in vector notation

$$\frac{1}{x - Q} = L(x) - \sum_{i=1}^{d_2} \psi_\infty^{(i)}(x)\hat{\phi}_\infty^{(i)t}(x) = R(x) + \psi_\infty(x)\hat{\phi}_\infty^t(x). \tag{3.29}$$

This implies

$$L(x) - R(x) = \sum_{i=0}^{d_2} \psi_\infty^{(i)}(x)\hat{\phi}_\infty^{(i)t}(x) = \Psi_\infty \hat{\Phi}_\infty^t \tag{3.30}$$

which is a matrix of rank $d_2 + 1$.

Since R is lower triangular, we have, for $n \leqslant m$,

$$L_{nm}(x) = \sum_{i=0}^{d_2} \psi_n^{(i)}(x)\hat{\phi}_m^{(i)}(x) \qquad \text{if} \quad n \leqslant m, \tag{3.31}$$

and since L is upper triangular, we have, for $n \geqslant m - d_2 + 1$,

$$R_{nm}(x) = -\sum_{i=0}^{d_2} \psi_n^{(i)}(x)\hat{\phi}_m^{(i)}(x) \qquad \text{if} \quad n \geqslant m - d_2 + 1. \tag{3.32}$$

Note that if $n \leqslant m \leqslant n + d_2 - 1$, we have

$$\sum_{i=0}^{d_2} \psi_n^{(i)}(x)\hat{\phi}_m^{(i)}(x) = 0 \qquad \text{if} \quad n \leqslant m \leqslant n + d_2 - 1, \tag{3.33}$$

and thus

$$\left(\frac{1}{x-Q}\right)_{nm} = \begin{cases} \psi_n(x)\hat{\phi}_m(x) & \text{if} \quad n \leqslant m \\ -\sum_{i=1}^{d_2} \psi_n^{(i)}(x)\hat{\phi}_m^{(i)}(x) & \text{if} \quad n \geqslant m - d_2 + 1. \end{cases} \tag{3.34}$$

4. Kernels

The kernels are the building blocks of non-mixed expectation values [2, 16], and we need them because they will appear in mixed expectation values as well. Thus, we recall their definitions and some properties below.

4.1. Definition of the kernels

4.1.1. K kernels. We introduce the kernel

$$K_n(x, y) := \sum_{m=0}^{n-1} \phi_m(y)\psi_m(x) = \phi_\infty^t(y)\Pi_{n-1}\psi_\infty(x) \tag{4.1}$$

and more generally

$$\begin{aligned} K_n^{(i,j)}(x, y) &:= \sum_{m=0}^{n-1} \phi_m^{(i)}(y)\psi_m^{(j)}(x) + K_0^{(i,j)}(x, y) \\ &= \phi_\infty^{(i)t}(y)\Pi_{n-1}\psi_\infty^{(j)}(x) + K_0^{(i,j)}(x, y) \end{aligned} \tag{4.2}$$

where

$$K_0^{(0,0)} := 0 \tag{4.3}$$

$$K_0^{(i,0)}(x, y) := -\frac{1}{2\mathrm{i}\pi}\,\mathrm{e}^{-V_1(x)} \int_{\overline{\gamma}^{(i)}} \frac{1}{x - x'}\,\mathrm{e}^{V_1(x')}\,\mathrm{e}^{x'y}\,\mathrm{d}x' \tag{4.4}$$

$$K_0^{(0,j)}(x, y) := -\frac{1}{2\mathrm{i}\pi}\,\mathrm{e}^{-V_2(y)} \int_{\tilde{\overline{\gamma}}^{(j)}} \frac{1}{y - y'}\,\mathrm{e}^{V_2(y')}\,\mathrm{e}^{xy'}\,\mathrm{d}y' \tag{4.5}$$

$$\begin{aligned} K_0^{(i,j)}(x, y) &:= \frac{1}{(2\mathrm{i}\pi)^2} \int_{\tilde{\overline{\gamma}}^{(i)}} \mathrm{d}x'' \int_{\overline{\gamma}^{(j)}} \mathrm{d}y'' \int_\Gamma \mathrm{d}\mu(x', y')\,\mathrm{e}^{V_1(x'')}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{xy''+x''y} \\ &\quad \times \left(\frac{1}{(x-x')(x-x'')}\frac{V_2'(y') - V_2'(y'')}{y'-y''} + \frac{1}{(y-y')(y-y'')}\frac{V_1'(x') - V_1'(x'')}{x'-x''}\right). \end{aligned} \tag{4.6}$$

4.1.2. J kernels. We also introduce the kernels

$$J_n(x, y) := J_0(x, y) - \sum_{m=0}^{n-1} \hat{\phi}_m(x)\hat{\psi}_m(y) := J_0(x, y) - \hat{\phi}_\infty(x)^t \Pi_{n-1} \hat{\psi}_\infty(y) \tag{4.7}$$

and more generally

$$J_n^{(i,j)}(x, y) := J_0^{(i,j)}(x, y) - \sum_{m=0}^{n-1} \hat{\phi}_m^{(i)}(x)\hat{\psi}_m^{(j)}(y) \tag{4.8}$$

where

$$J_0(x, y) := J_0^{(0,0)}(x, y) := \mathrm{e}^{V_1(x)}\, \mathrm{e}^{V_2(y)} \int_\Gamma \frac{1}{x - x'} \frac{1}{y - y'}\, \mathrm{d}\mu(x', y') \tag{4.9}$$

$$J_0^{(0,i)}(x, y) := \mathrm{e}^{V_1(x)} \int_{\gamma^{(i)}} \frac{1}{x - x'}\, \mathrm{e}^{-V_1(x')}\, \mathrm{e}^{-x'y}\, \mathrm{d}x' \tag{4.10}$$

$$J_0^{(i,0)}(x, y) := \mathrm{e}^{V_2(y)} \int_{\tilde{\gamma}^{(i)}} \frac{1}{y - y'}\, \mathrm{e}^{-V_2(y')}\, \mathrm{e}^{-xy'} \mathrm{d}y' \tag{4.11}$$

$$J_0^{(i,j)}(x, y) := \mathrm{e}^{-xy}. \tag{4.12}$$

Sometimes one writes abusively $J_n(x, y) = \sum_{m=n}^{\infty} \hat{\phi}_m(x)\hat{\psi}_m(y) = \hat{\phi}_\infty^t \Pi^n \hat{\psi}_\infty$.

4.1.3. H kernels. We also introduce the kernels

$$H_n^{(ij)}(x_1, x_2) := \sum_{m=0}^{n-1} \hat{\phi}_m^{(i)}(x_1)\psi_m^{(j)}(x_2) + H_0^{(ij)}(x_1, x_2) \tag{4.13}$$

where

$$H_0^{(i,j)}(x_1, x_2) := \frac{1}{x_2 - x_1} \frac{\mathrm{e}^{V_1(x_1) - V_1(x_2)}}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}^{(j)}} \mathrm{d}y \int_{\tilde{\gamma}^{(i)}} \mathrm{d}y' \frac{V_2'(y) - V_2'(y')}{y - y'} \frac{\omega(x_1, y')}{\omega(x_2, y)} \tag{4.14}$$

$$H_0^{(0,j)}(x_1, x_2) := \frac{\mathrm{e}^{V_1(x_1) - V_1(x_2)}}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}^{(j)}} \mathrm{d}y \int_\Gamma \mathrm{d}x'\, \mathrm{d}y' \frac{V_2'(y) - V_2'(y')}{y - y'} \frac{1}{x' - x_1} \frac{1}{x' - x_2} \frac{\omega(x', y')}{\omega(x_2, y)} \tag{4.15}$$

$$H_0^{(i,0)}(x_1, x_2) := 0 \tag{4.16}$$

$$H_0^{(0,0)}(x_1, x_2) := \frac{1}{x_2 - x_1}\, \mathrm{e}^{V_1(x_1) - V_1(x_2)}. \tag{4.17}$$

Note that at $x_1 = x_2$ we have

$$\lim_{x_1 \to x_2} (x_2 - x_1) H_n^{(i,j)}(x_1, x_2) = \delta_{i,j}. \tag{4.18}$$

4.1.4. $\tilde{H}$ kernels. Similarly,

$$\tilde{H}_n^{(ij)}(y, y') := \sum_{m=0}^{n-1} q_m^{(i)}(y)\hat{p}_m^{(j)}(y') + \tilde{H}_0^{(ij)}(y, y'). \tag{4.19}$$

4.2. Christoffel–Darboux matrices

We define the Christoffel–Darboux matrices [9]:

$$A_n := [\Pi_{n-1}, Q] = \gamma_n \mathbf{e}_{n-1}\mathbf{e}_n^t - (1 - \Pi_{n-1}) Q \Pi_{n-1}, \tag{4.20}$$

$$B_n := [\Pi_{n-1}, P] = \gamma_n \mathbf{e}_{n-1}\mathbf{e}_n^t - (1 - \Pi_{n-1}) P \Pi_{n-1}. \tag{4.21}$$

Note that since Q (resp. P) is finite band, A_n (resp. B_n) is non-vanishing only in a sub-block of size $d_2 + 1 \times d_2 + 1$ (resp. $d_1 + 1 \times d_1 + 1$), i.e.,

$$(A_n)_{ij} \neq 0 \qquad \text{iff} \quad n - d_2 \leqslant i \leqslant n \qquad \text{and} \qquad n - 1 \leqslant j \leqslant n + d_2 - 1 \tag{4.22}$$

$$(B_n)_{ij} \neq 0 \qquad \text{iff} \quad n - d_1 \leqslant i \leqslant n \qquad \text{and} \qquad n - 1 \leqslant j \leqslant n + d_1 - 1. \tag{4.23}$$

We say that A_n and B_n are '*small matrices*', i.e. their size is not growing with n.

4.3. Christoffel–Darboux theorems

Using the recursion relations equation equation (2.39)–(2.56), we have the following Christoffel–Darboux theorems [9, 10]:

Theorem 4.1.

$$(x_2 - x_1) H_n^{(ij)}(x_1, x_2) = \hat{\phi}_\infty^{(i)t}(x_1) A_n \psi_\infty^{(j)}(x_2) \tag{4.24}$$

$$\left(\partial_{x_1} + \partial_{x_2}\right) H_n^{(ij)}(x_1, x_2) = -\hat{\phi}_\infty^{(i)t}(x_1) B_n^t \psi_\infty^{(j)}(x_2) \tag{4.25}$$

$$(x - \partial/\partial y) K_n^{(ij)}(x, y) = \phi_\infty^{(i)t}(y) A_n \psi_\infty^{(j)}(x) \tag{4.26}$$

$$(y - \partial/\partial x) K_n^{(ij)}(x, y) = \phi_\infty^{(i)t}(y) B_n^t \psi_\infty^{(j)}(x) \tag{4.27}$$

$$(x + \partial/\partial y) J_n(x, y) = \hat{\phi}_\infty^{(i)t}(y) A_n \hat{\psi}_\infty^{(j)}(x) \tag{4.28}$$

$$(y + \partial/\partial x) J_n(x, y) = \hat{\phi}_\infty^{(i)t}(y) B_n^t \hat{\psi}_\infty^{(j)}(x). \tag{4.29}$$

In particular, at $x_1 = x_2 = x$ in equation (4.24), and using equation (4.18), we recover the duality of [9]:

$$\hat{\phi}_\infty^{(i)t}(x) A_n \psi_\infty^{(j)}(x) = \delta_{i,j} \tag{4.30}$$

and we find

$$H_n^{(ij)}(x, x) = \hat{\phi}_\infty^{(i)t}(x) A_n \psi_\infty^{'(j)}(x). \tag{4.31}$$

5. Determinantal formula for the orthogonal polynomials and kernels

5.1. Determinantal formulae

The following formulae are very useful to express kernels or polynomials, as well as the Cauchy transforms, as $n \times n$ determinants. The proofs are given in appendix B.

1. Determinant of $\Pi_{n-1}(x - Q)\Pi_{n-1}$:

$$\det(\Pi_{n-1}(x - Q)\Pi_{n-1}) = p_n(x), \qquad \det(\Pi_{n-1}(y - P)\Pi_{n-1}) = q_n(y). \tag{5.1}$$

2. Determinant of $\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}$:

$$\det\left(\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\right)=\frac{\mathrm{e}^{-V_1(x)}}{\sqrt{h_{n-1}}}\hat{\phi}_{n-1}(x) \tag{5.2}$$

$$\det\left(\Pi_{n-1}\frac{1}{y-P}\Pi_{n-1}\right)=\frac{\mathrm{e}^{-V_2(y)}}{\sqrt{h_{n-1}}}\hat{\psi}_{n-1}(y). \tag{5.3}$$

3. Kernel K_n:

$$\det(\Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1})=h_n\,\mathrm{e}^{V_1(x)+V_2(y)}K_{n+1}(x,y). \tag{5.4}$$

4. Kernel J_n:

$$\det\left(\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1}\right)=\frac{\mathrm{e}^{-V_1(x)-V_2(y)}}{h_{n-1}}J_{n-1}(x,y). \tag{5.5}$$

5.2. Inverses

It is also useful to compute the inverses (i.e. all minors) of the previous matrices. The proofs are given in appendix B. The following formula give inverses of $n\times n$ matrices.

1. Inverse of $\Pi_{n-1}(x-Q)\Pi_{n-1}$:

$$(\Pi_{n-1}(x-Q)\Pi_{n-1})^{-1}=\Pi_{n-1}\left(1-\frac{\psi_\infty\mathbf{e}_n^t}{\psi_n}\right)\Pi_n R\Pi_{n-1}. \tag{5.6}$$

2. Inverse of $\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}$:

$$\left(\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\right)^{-1}=\Pi_{n-1}\left(1+\frac{\mathbf{e}_{n-1}\hat{\phi}_\infty^t}{\hat{\phi}_{n-1}}\Pi^n\right)(x-Q)\Pi_{n-1}. \tag{5.7}$$

3. Inverse of $\Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1}$:

$$(\Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1})^{-1}=\Pi_{n-1}\tilde{R}^t\Pi_n\left(1-\frac{1}{K_{n+1}}\psi_\infty\phi_\infty^t\right)\Pi_n R\Pi_{n-1}. \tag{5.8}$$

4. Inverse of $\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1}$:

$$\begin{aligned}\left(\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1}\right)^{-1}&=\Pi_{n-1}(y-P^t)\\&\times\left(\Pi_{n-2}+\frac{1}{J_{n-1}}(1-\Pi_{n-2})\hat{\psi}_\infty\hat{\phi}_\infty^t(1-\Pi_{n-2})\right)(x-Q)\Pi_{n-1}.\end{aligned} \tag{5.9}$$

5. Equivalent formula for $\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1}$:

$$\begin{aligned}\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1}&=\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\Pi_{n-1}+J_n\Pi_{n-1}\psi_\infty\phi_\infty^t\Pi_{n-1}\\&=\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-2}\frac{1}{y-P^t}\Pi_{n-1}+J_{n-1}\Pi_{n-1}\psi_\infty\phi_\infty^t\Pi_{n-1}.\end{aligned} \tag{5.10}$$

6. Windows

The Christoffel–Darboux theorems of section 4.3 show that the kernels can be computed in terms of only ψ_m with $n-d_2 \leqslant m \leqslant n$ and $\hat{\phi}_m$ with $n-1 \leqslant m \leqslant n+d_2-1$ (resp. ϕ_m with $n-d_1 \leqslant m \leqslant n$ and $\hat{\psi}_m$ with $n-1 \leqslant m \leqslant n+d_1-1$). We thus introduce the following vectors, called 'windows' [9–11]:

$$\vec{\psi}_n(x) = \left(\psi_{n-d_2}, \psi_{n-d_2+1}, \ldots, \psi_{n-1}, \psi_n\right)^t = \Pi_n^{n-d_2}\psi_\infty \tag{6.1}$$

and similarly

$$\vec{\phi}_n(y) = \Pi_n^{n-d_1}\phi_\infty, \qquad \vec{\hat{\psi}}_n(y) = \Pi_{n+d_1-1}^{n-1}\hat{\psi}_\infty, \qquad \vec{\hat{\phi}}_n(y) = \Pi_{n+d_2-1}^{n-1}\hat{\phi}_\infty. \tag{6.2}$$

We also introduce the following matrices, called 'windows' [9–11]:

$$\Psi_n(x)_{mi} := \psi_m^{(i)}(x), \qquad m = n-d_2, \ldots, n, \qquad i = 0, \ldots, d_2 \tag{6.3}$$

$$\Phi_n(y)_{mi} := \phi_m^{(i)}(y), \qquad m = n-d_1, \ldots, n, \qquad i = 0, \ldots, d_1 \tag{6.4}$$

$$\hat{\Psi}_n(y)_{mi} := \hat{\psi}_m^{(i)}(y), \qquad m = n-1, \ldots, n+d_1-1, \qquad i = 0, \ldots, d_1 \tag{6.5}$$

$$\hat{\Phi}_n(x)_{mi} := \hat{\phi}_m^{(i)}(x), \qquad m = n-1, \ldots, n+d_2-1, \qquad i = 0, \ldots, d_2. \tag{6.6}$$

The relationship equation (4.30) implies the duality (found in [9])

$$\hat{\Phi}_n^t(x)A_n\Psi_n(x) = \mathrm{Id}_{d_2+1} \tag{6.7}$$

and similarly

$$\Phi_n^t(y)B_n^t\hat{\Psi}_n(y) = \mathrm{Id}_{d_1+1}. \tag{6.8}$$

Note that (combining equations (3.30) and (6.7))

$$\Psi_\infty(x) = (L(x) - R(x))A_n\Psi_n(x). \tag{6.9}$$

The matrix

$$F_n(x) := (L(x) - R(x))A_n \tag{6.10}$$

is the so-called Folding matrix of [11]. Its property is to fold any operator acting on Ψ_∞ into an operator acting on the window only. For any finite-band operator $\hat{O}$, we have

$$\Pi_n^{n-d_2}\hat{O}\Psi_\infty(x) = \left(\Pi_n^{n-d_2}\hat{O}F_n(x)\right)\Psi_n(x) \tag{6.11}$$

where $\Pi_n^{n-d_2}\hat{O}F_n(x)$ is a square matrix of size $d_2+1\times d_2+1$. When acting on the window Ψ_n, it gives the same result as the operator $\hat{O}$ acting on the infinite matrix Ψ_∞.

7. Mixed correlation function

We now arrive at the main results of this paper, which concerns the computation of the mixed correlation function

$$W_n(x, y) = 1 + \left\langle \mathrm{Tr}\,\frac{1}{x-M_1}\frac{1}{y-M_2}\right\rangle, \tag{7.1}$$

where M_1 and M_2 are in the ensemble $H_n \times H_n(\Gamma)$ of remark 2.1, and explained in [18], with the measure

$$\frac{1}{Z}\exp(-\mathrm{Tr}\,(V_1(M_1) + V_2(M_2) + M_1M_2))\,\mathrm{d}M_1\,\mathrm{d}M_2. \tag{7.2}$$

and where Z is the normalization constant, called partition function:

$$Z = \int_{H_n \times H_n(\Gamma)} \exp(-\mathrm{Tr}\,(V_1(M_1) + V_2(M_2) + M_1 M_2))\,\mathrm{d}M_1\,\mathrm{d}M_2. \tag{7.3}$$

We recall that $H_n \times H_n(\Gamma)$ is the set of normal matrices (i.e. $[M_1, M_1^\dagger] = 0 = [M_2, M_2^\dagger]$) with pairs of eigenvalues constrained to be on Γ, see [18] for more details. In the case $\Gamma = \mathbf{R} \times \mathbf{R}$, M_1 and M_2 are Hermitian matrices of size n.

The formula of [8] reads

$$W_n(x, y) = \det_n \left(\mathrm{Id}_n + \Pi_{n-1} \frac{1}{x - Q} \frac{1}{y - P^t} \Pi_{n-1} \right) \tag{7.4}$$

which is an $n \times n$ determinant, and is thus not convenient for large n computations and for many other applications.

The purpose of this section is to write it in terms of a determinant of size $d_2 + 1$ or $d_1 + 1$.

7.1. Formula for W_n

We introduce the following lower triangular matrices[2] of size $d_1 + 1$ (resp. $d_2 + 1$):

$$\begin{aligned} U_n(x, y) &:= -\frac{y + V_1'(x)}{\gamma_n} \mathbf{e}_n \mathbf{e}_{n-1}^t - \Pi_n \frac{V_1'(x) - V_1'(Q)}{x - Q} \Pi^{n-1} \\ \tilde{U}_n(x, y) &:= -\frac{x + V_2'(y)}{\gamma_n} \mathbf{e}_n \mathbf{e}_{n-1}^t - \Pi_n \frac{V_2'(y) - V_2'(P)}{y - P} \Pi^{n-1}. \end{aligned} \tag{7.5}$$

Many of their properties are described in appendix C.

The following theorem is proved in appendix D:

Theorem 7.1.

$$\begin{aligned} W_n(x, y) &= \gamma_n^2 K_{n+1}(x, y) J_{n-1}(x, y) \det \left(\mathrm{Id}_n - \Pi_n \left(1 - \frac{\psi_\infty \phi_\infty^t}{K_{n+1}} \right) U_n \left(1 - \frac{\hat{\psi}_\infty \hat{\phi}_\infty^t}{J_{n-1}} \right) \tilde{U}_n^t \right) \\ &= \gamma_n^2 K_{n+1}(x, y) J_{n-1}(x, y) \det \left(\mathrm{Id}_{d_2+1} - \Pi_n^{n-d_2} \left(1 - \frac{\vec{\psi}_n \vec{\phi}_n^t}{K_{n+1}} \right) U_n \left(1 - \frac{\hat{\vec{\psi}}_n \hat{\vec{\phi}}_n^t}{J_{n-1}} \right) \tilde{U}_n^t \right) \\ &= \gamma_n^2 K_{n+1}(x, y) J_{n-1}(x, y) \det \left(\mathrm{Id}_{d_1+1} - \Pi_n^{n-d_1} \left(1 - \frac{\hat{\vec{\psi}}_n \hat{\vec{\phi}}_n^t}{J_{n-1}} \right) \tilde{U}_n^t \left(1 - \frac{\vec{\psi}_n \vec{\phi}_n^t}{K_{n+1}} \right) U_n \right). \end{aligned} \tag{7.6}$$

or also

$$\boxed{\begin{aligned} W_n(x, y) = \gamma_n^2 &\det(\mathrm{Id}_{n+1} - U_n(x, y) \tilde{U}_n(x, y)^t) \\ &\times \left(\left(J_{n+d_2} + \hat{\vec{\phi}}_n^t \frac{1}{1 - \tilde{U}_n^t U_n} \hat{\vec{\psi}}_n \right) \left(K_{n-d_2} + \vec{\phi}_n^t \frac{1}{1 - U_n \tilde{U}_n^t} \vec{\psi}_n \right) \right. \\ &\left. - \left(\hat{\vec{\phi}}_n^t \tilde{U}_n^t \frac{1}{1 - U_n \tilde{U}_n^t} \vec{\psi}_n \right) \left(\vec{\phi}_n^t \frac{1}{1 - U_n \tilde{U}_n^t} U_n \hat{\vec{\psi}}_n \right) \right). \end{aligned}} \tag{7.7}$$

Note that those formulae involve only functions which are within the windows. The determinants are in fact of size $\min(d_2 + 1, d_1 + 1)$.

The proof is given in appendix D.

[2] They are infinite matrices, with only a non-vanishing sub-block of size $d_1 + 1$ (resp. $d_2 + 1$).

Theorem 7.2.

$$W_n(x, y) = -\tilde{t} K_{n-d_2}(x, y) J_{n+d_2}(x, y) \det(y - M(x, y)) \tag{7.8}$$

where M is the $(d_2 + 1) \times (d_2 + 1)$ *matrix:*

$$M_{ij} = H_n^{(ij)}(x, x) + \frac{\delta_{i0}}{K_{n-d_2}(x, y)}(y - \partial_x) K_{n-d_2}^{(0j)}(x, y) + \frac{\delta_{j0}}{J_{n+d_2}(x, y)}(y + \partial_x) J_{n+d_2}^{(i0)}(x, y). \tag{7.9}$$

Proof. We use expressions of appendix F and lemma 1.2. □

7.2. Recursion $W_{n+1} - W_n$

We introduce the following lower triangular matrices of size d_1 (resp. d_2):

$$\mathcal{W}_n(x) = \Pi_n \frac{V_1'(x) - V_1'(Q)}{x - Q} \Pi^n, \qquad \tilde{\mathcal{W}}_n(y) = \Pi_n \frac{V_2'(y) - V_2'(P)}{y - P} \Pi^n \tag{7.10}$$

The following theorem is proved in appendix E.

Theorem 7.3.

$$W_{n+1}(x, y) - W_n(x, y) = K_{n+1} J_n \det\left(1 - \Pi_n\left(1 - \frac{\psi_\infty \phi_\infty^t}{K_{n+1}}\right)\mathcal{W}_n\left(1 - \frac{\hat{\psi}_\infty \hat{\phi}_\infty^t}{J_n}\right)\tilde{\mathcal{W}}_n^t\right). \tag{7.11}$$

i.e.

$$\boxed{\begin{aligned} W_{n+1}(x, y) - W_n(x, y) &= \det\left(1 - \mathcal{W}_n \tilde{\mathcal{W}}_n^t\right) \\ &\quad \times \left[\left(J_n + \left(\hat{\phi}_\infty^t \mathcal{W}_n^t \frac{1}{1 - \mathcal{W}_n \tilde{\mathcal{W}}_n^t} \tilde{\mathcal{W}}_n \hat{\psi}_\infty\right)\right)\left(\phi_\infty^t \frac{1}{1 - \mathcal{W}_n \tilde{\mathcal{W}}_n^t} \psi_\infty\right)\right. \\ &\quad \left. - \left(\phi_\infty^t \frac{1}{1 - \mathcal{W}_n \tilde{\mathcal{W}}_n^t} \mathcal{W}_n \hat{\psi}_\infty\right)\left(\hat{\phi}_\infty^t \tilde{\mathcal{W}}_n^t \frac{1}{1 - \mathcal{W}_n \tilde{\mathcal{W}}_n^t} \psi_\infty\right)\right], \end{aligned}} \tag{7.12}$$

i.e. $W_{n+1} - W_n$ *actually involves the computation of a determinant and inverse of a matrix of size* $\min(d_1, d_2)$.

The proof is given in appendix E.

8. Application: differential systems and spectral curve

An important observation of [9] and [10, 29, 32] is that windows of consecutive bi-orthogonal polynomials satisfy some integrable differential systems and a Riemann–Hilbert problem. An explicit representation of those differential systems was found in [11]. Here, thanks to the result for the mixed correlation function, we are able to give new representations of those systems and compute explicitly their spectral curve.

8.1. Differential systems

We define the following $(d_2 + 1) \times (d_2 + 1)$ and $(d_1 + 1) \times (d_1 + 1)$ matrices:

$$\mathcal{D}_n(x) := \Psi_n'(x) \Psi_n^{-1}(x), \qquad \hat{\mathcal{D}}_n(x) := -\hat{\Phi}_n'(x) \hat{\Phi}_n^{-1}(x) \tag{8.1}$$

$$\tilde{\mathcal{D}}_n(y) := \Phi_n'(y) \Phi_n^{-1}(y), \qquad \hat{\tilde{\mathcal{D}}}_n(y) := -\hat{\Psi}_n'(y) \hat{\Psi}_n^{-1}(y). \tag{8.2}$$

Those matrices are square matrices of the size of the corresponding window and they have polynomial entries. They give some ODE's for the windows:

$$\Psi'_n(x) = \mathcal{D}_n(x)\Psi_n(x). \tag{8.3}$$

It was found in [9] that they enjoy some duality relations (which merely come from duality equation (6.7)):

$$\hat{\mathcal{D}}^t_n(x)A_n = A_n\mathcal{D}_n(x), \qquad \hat{\tilde{\mathcal{D}}}^t_n(y)B_n = B_n\tilde{\mathcal{D}}_n(y). \tag{8.4}$$

Theorem 8.1. *We have the following equivalent expressions for $\mathcal{D}$:*

$$\boxed{\begin{array}{l}\mathcal{D}_n(x) = \Pi_n^{n-d_2}P^t(L(x)-R(x))A_n\\ y-\mathcal{D}_n(x) = \left(\tilde{U}_n^{t-1}(x,y) - \Pi_n^{n-d_2}U_n(x,y)\right)A_n\\ \left(\Psi_n^{-1}(x)\mathcal{D}_n(x)\Psi_n(x)\right)_{ij} = H^{(ij)}(x,x).\end{array}} \tag{8.5}$$

And the matrix elements of $y - \mathcal{D}_n(x)$ are $(d_1+1)\times(d_1+1)$ determinants given by $(0 \leqslant k, l \leqslant d_2)$:

- *If $k \geqslant d_1$*

$$(y-\mathcal{D}_n(x))_{n-k,n-l} = \frac{1}{\prod_{m=1}^{d_1} Q_{n-m,n-m-d_2}} \times \det\begin{pmatrix}\mathbf{e}^t_{n-k}(y-P^t)\Pi^{n-d_1-d_2}_{n-d_2-1} & \mathbf{e}^t_{n-k}(y-P^t)\,\mathbf{e}_{n-l}\\ \Pi^{n-d_1}_{n-1}(x-Q)\Pi^{n-d_1-d_2}_{n-d_2-1} & \Pi^{n-d_1}_{n-1}(x-Q)\,\mathbf{e}_{n-l}\end{pmatrix}. \tag{8.6}$$

- *If $k < d_1$*

$$(y-\mathcal{D}_n(x))_{n-k,n-l} = \frac{(-1)^{d_1-k}}{\prod_{m=0}^{d_1-k}\gamma_{n+m-1}\prod_{m=1}^{k} Q_{n-m,n-m-d_2}}$$
$$\times \det\begin{pmatrix}\mathbf{e}^t_{n-k}(y-P^t)\Pi^{n-d_2-k}_{n-d_2-1} & \mathbf{e}^t_{n-k}(y-P^t)\,\mathbf{e}_{n-l} & \mathbf{e}^t_{n-k}(y-P^t)\Pi^{n+1}_{n+d_1-k}\\ \Pi^{n-k}_{n+d_1-k-1}(x-Q)\Pi^{n-d_2-k}_{n-d_2-1} & \Pi^{n-k}_{n+d_1-k-1}(x-Q)\,\mathbf{e}_{n-l} & \Pi^{n-k}_{n+d_1-k-1}(x-Q)\Pi^{n+1}_{n+d_1-k}\end{pmatrix}. \tag{8.7}$$

The first equality of equation (8.5) was found in [11], it comes from the Folding matrix equation (6.10): $\Psi'_n = \Pi_n^{n-d_2}\Psi'_\infty = \Pi_n^{n-d_2}P^t\Psi_\infty = \Pi_n^{n-d_2}P^tF_n(x)\Psi_n$.

The second equality is a rewriting of the first one using the matrices U_n and $\tilde{U}_n$, see appendix C.

The third equality is a mere rewriting of the definition of $\mathcal{D}$, using the duality equation (4.30).

The last representation is obtained by using the first formula and computing the folding matrix by inverting the linear problem $\Psi_\infty = F_n\Psi_n$ in the kernel of $x - Q$.

8.2. Spectral curve

It was proven in [9] that all those differential systems share the same spectral curve. We define

$$\begin{aligned}\mathcal{E}_n(x,y) &:= \tilde{t}\det(y-\mathcal{D}_n(x)) = \tilde{t}\det(y-\hat{\mathcal{D}}_n(x))\\ &= t\det(x-\tilde{\mathcal{D}}_n(y)) = t\det(x-\hat{\tilde{\mathcal{D}}}_n(y)).\end{aligned} \tag{8.8}$$

We are going to give here several equivalent formulae for computing $\mathcal{E}$.

Theorem 8.2. *We have*

$$\boxed{\mathcal{E}_n(x,y) = -\gamma_n^2 \det\left(1 - U_n \tilde{U}_n^t\right)} \tag{8.9}$$

and

$$\mathcal{E}_{n+1}(x,y) - \mathcal{E}_n(x,y) = -\det\left(1 - \mathcal{W}_n \tilde{\mathcal{W}}_n^t\right). \tag{8.10}$$

Proof. For the first expression, use the second expression in theorem 8.1. The proof of the recursion formula is found in appendix H. □

Then, we introduce the following lemma, which consists in taking the polynomial part at large y of formula equation (7.7):

Lemma 8.1.

$$\boxed{\operatorname*{Pol}_{y\to\infty} (V_2'(y) + x) W_n(x,y) = \tilde{t} \det_{i,j\neq 0} (y - H^{(ij)})} \tag{8.11}$$

i.e.

$$V_2'(y) + x + \left\langle \operatorname{Tr} \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle = \tilde{t} \det_{i,j\neq 0} (y - H^{(ij)}). \tag{8.12}$$

Note that this formula is the finite n counterpart of what was found in the formal large n expansion in [14], namely theorem 3.1, equation (3.3) of [14].

Proof. This lemma is proved in appendix F.5. □

Using this lemma, taking the polynomial part in x at large x, we prove the following theorem, which was conjectured by Marco Bertola [5]:

Theorem 8.3. *Proof of Bertola's conjecture:*

$$\operatorname*{Pol}_{x\to\infty} \operatorname*{Pol}_{y\to\infty} (V_1'(x) + y)(V_2'(y) + x) W_n(x,y) = 2n + \mathcal{E}_n(x,y) \tag{8.13}$$

i.e.

$$\boxed{\mathcal{E}_n(x,y) = (V_1'(x) + y)(V_2'(y) + x) + \left\langle \operatorname{Tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right\rangle - n.} \tag{8.14}$$

Marco Bertola has proved it for potentials of degree $\max(d_1, d_2) \leqslant 5$, which is quite remarkable, and also for the smallest values of n [5].

This theorem is also the finite n counterpart of theorem 3.1 of [14]. Note that the spectral curve for finite n is the same as the algebraic curve found from large n considerations [14]. This fact has important consequences, some of which are described below in the next section.

Proof. The proof is in appendix G. □

For completeness, we also write a formula for the spectral curve, due to Jacques Hurtubise:

Theorem 8.4. *Hurtubise formula [26]*

$$\mathcal{E}_n(x,y) = -\frac{\gamma_n}{\prod_{j=n-d_2}^{n+d_1} \gamma_j} \det \begin{pmatrix} \Pi_n^{n-d_2}(P^t - y) \\ \Pi_{n+d_1-1}^{n-1}(Q - x) \end{pmatrix} \tag{8.15}$$

which simply amounts to say that if $\mathcal{E}_n(x,y) = 0$, *there must exist some functions* $\psi_{n-d_2-1}, \ldots, \psi_{n+d_1}$, *such that we have simultaneously* $y\psi_i(x) = \psi_i'(x) = \sum_i P_{ji}\psi_j(x)$ *for* $i = n - d_2, \ldots, n$ *and* $x\psi_i(x) = \sum_i Q_{ij}\psi_j(x)$ *for* $i = n-1, \ldots, n+d_1-1$, *i.e. the vector* $(\psi_{n-d_2-1}, \ldots, \psi_{n+d_1})^t$ *is in the kernel of the matrix above.*

9. Examples

9.1. Gaussian case, Ginibre polynomials

Consider $V_1(x) = V_2(y) = 0$, which is Ginibre's ensemble. It is well known that the bi-orthogonal polynomials are monomials [24]:

$$p_n(x) = x^n, \qquad q_n(y) = y^n, \qquad h_n = n!\pi, \tag{9.1}$$

and the Cauchy transforms are

$$\hat{p}_n(y) = \frac{\pi n!}{y^{n+1}}, \qquad \hat{q}_n(x) = \frac{\pi n!}{x^{n+1}}. \tag{9.2}$$

We have

$$Q_{nm} = P_{nm} = \sqrt{n+1}\delta_{m,n+1}, \qquad \gamma_n = \sqrt{n}. \tag{9.3}$$

We find that for $n > m$

$$R_{nm}(x) = -\psi_n(x)\hat{\phi}_m(x), \tag{9.4}$$

and $\frac{1}{x-Q}$ is an upper triangular matrix.

In that case, theorem 7.1 becomes

$$\begin{aligned} W_n(x, y) &= 1 + nJ_{n-1}(x, y)K_n(x, y) - xyJ_n(x, y)K_{n-1}(x, y) \\ &= nK_{n+1}J_{n-1} - xyK_nJ_n, \end{aligned} \tag{9.5}$$

and theorem (7.3) with $\mathcal{W}_n = 0$ becomes

$$W_{n+1} - W_n = K_{n+1}J_n. \tag{9.6}$$

9.2. Gaussian elliptical case

Consider $V_1'(x) = tx$, $V_2'(y) = \tilde{t}y$, we write

$$\delta = 1 - t\tilde{t}. \tag{9.7}$$

The bi-orthogonal polynomials are rescaled Hermite polynomials [33], and we have

$$\gamma_n = Q_{n-1,n} = P_{n-1,n} = \sqrt{\frac{n}{\delta}} \tag{9.8}$$

$$Q_{n,n-1} = -\tilde{t}\gamma_n, \qquad P_{n,n-1} = -t\gamma_n. \tag{9.9}$$

The Christoffel–Darboux matrices are

$$A_n = \gamma_n \begin{pmatrix} 0 & 1 \\ \tilde{t} & 0 \end{pmatrix}, \qquad B_n = \gamma_n \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}. \tag{9.10}$$

The matrices U_n, $\tilde{U}_n$ are

$$U_n(x, y) = -\begin{pmatrix} t & 0 \\ \frac{y+tx}{\gamma_n} & t \end{pmatrix}, \qquad \tilde{U}_n^t(x, y) = -\begin{pmatrix} \tilde{t} & \frac{x+\tilde{t}y}{\gamma_n} \\ 0 & \tilde{t} \end{pmatrix} \tag{9.11}$$

and the matrices $\mathcal{W}_n$ and $\tilde{\mathcal{W}}_n$ are of dimension 1:

$$\mathcal{W}(x) = t, \qquad \tilde{\mathcal{W}}(y) = \tilde{t}. \tag{9.12}$$

The differential systems are

$$\mathcal{D}_n(x) = \begin{pmatrix} -\frac{x}{\tilde t} & \frac{\delta}{\tilde t}\gamma_n \\ \delta\gamma_n & -tx \end{pmatrix}, \qquad \overline{\mathcal{D}}_n(x) = \begin{pmatrix} -tx & \delta\gamma_n \\ \frac{\delta}{\tilde t}\gamma_n & -\frac{x}{\tilde t} \end{pmatrix} \tag{9.13}$$

$$\tilde{\mathcal{D}}_n(y) = \begin{pmatrix} -\frac{y}{t} & \frac{\delta}{t}\gamma_n \\ \delta\gamma_n & -\tilde t y \end{pmatrix}, \qquad \overline{\tilde{\mathcal{D}}}_n(y) = \begin{pmatrix} -\tilde t y & \delta\gamma_n \\ \frac{\delta}{t}\gamma_n & -\frac{y}{t} \end{pmatrix}. \tag{9.14}$$

thus the spectral curve is

$$\mathcal{E}_n(x,y) = \tilde t \det(y - \mathcal{D}_n(x)) = (x + \tilde t y)(y + tx) - \delta n = -\gamma_n^2 \det\left(1 - U_n \tilde U_n^t\right). \tag{9.15}$$

Theorem 7.3 gives

$$W_{n+1} - W_n = J_n K_{n+1} - t\tilde t J_{n+1} K_n. \tag{9.16}$$

10. Tau function

Theorem 8.3 implies for $k \leqslant d_1$ and $l \leqslant d_2$

$$\left\langle \operatorname{Tr} M_1^k M_2^l \right\rangle = \operatorname*{Res}_{x\to\infty} \operatorname*{Res}_{y\to\infty} \frac{\mathcal{E}_n(x,y) x^k y^l}{V_1'(x) V_2'(y)} \, \mathrm{d}x \, \mathrm{d}y. \tag{10.1}$$

Thus, if we introduce the matrix integral

$$\tilde Z = \int \mathrm{d}M_1 \, \mathrm{d}M_2 \, \exp\left(-\left[\sum_{k,l} t_{k,l} \operatorname{Tr} M_1^k M_2^l\right]\right) \tag{10.2}$$

we have

$$\operatorname*{Res}_{x\to\infty} \operatorname*{Res}_{y\to\infty} \frac{\mathcal{E}_n(x,y) x^k y^l}{V_1'(x) V_2'(y)} \, \mathrm{d}x \, \mathrm{d}y = -\left. \frac{\partial \ln \tilde Z}{\partial t_{k,l}} \right|_{t_{k,l}=0 \text{ if } kl>1}. \tag{10.3}$$

Equation (10.3) is similar to the tau function of [22, 27, 28, 36] generalized to the resonant case by [7], and thus, similarly to [12], we have proved that

$$\tau(\{t_{k,l}\}) = \tilde Z \qquad \text{for} \quad t_{k,l} = 0 \quad \text{if} \quad kl > 1. \tag{10.4}$$

This strongly suggests that the general setting of the isomonodromic problem should include all $t_{k,l}$'s.

It is easy to define bi-orthogonal polynomials with a weight of the form

$$\omega(x,y) = \exp\left(-\sum_{k,l} t_{k,l} x^k y^l\right)$$

and they are related to the so-called ensemble of normal complex-matrix integrals (see [38]):

$$Z_{\text{normal}} = \int_{N_n} \mathrm{d}M \exp\left(-\sum_{k,l} t_{k,l} \operatorname{Tr} M^k M^{\dagger l}\right) \tag{10.5}$$

where N_n is the set of complex matrices which commute with their adjoint $[M, M^\dagger] = 0$ (they have complex eigenvalues not constrained on any path). For normal matrices, we have

$$\left\langle \operatorname{Tr} \frac{1}{x - M} \frac{1}{y - M^\dagger} \right\rangle = \operatorname{Tr}\left(\Pi_{n-1} \frac{1}{y - P^t} \frac{1}{x - Q} \Pi_{n-1}\right) \tag{10.6}$$

so that our W_n is not the mixed correlation function for normal matrices.

Our $W_n(x,y)$ is in that model:

$$W_n(x,y) = \left\langle \det\left(1 + \frac{1}{x - M} \frac{1}{y - M^\dagger}\right)\right\rangle = \det\left(1 + \Pi_{n-1} \frac{1}{x - Q} \frac{1}{y - P^t} \Pi_{n-1}\right). \tag{10.7}$$

It is not clear yet why this function should be related to the spectral curve for the normal matrix model . . . .

11. Conclusion

In this paper, we have found a formula for computing the mixed correlation function as a $\min(d_1+1, d_2+1)$ determinant, instead of an $n \times n$ determinant. This kind of formula can be very useful for finding large n limits. Beside, we have found several expressions for the spectral curve, which can be useful for studying the integrability properties. In particular, we have proved the conjecture of Marco Bertola, which should open the route to other applications.

In this paper, we have computed the two-point mixed correlation function, however, for many applications to physics (BMN limit of ADS/CFT correspondence in string theory or boundary conformal field theory), one needs mixed correlation functions involving more than one trace and more than two matrices in a trace. The key element of the work of [8] was the use of Morozov's formula for two-point correlation functions of unitary integrals. Since then, some generalizations of Morozov's formula have been found for arbitrary correlation functions [21], and it seems natural to generalize the result of [8] and of the present paper. In that prospect, one could expect to understand the Bethe-ansatz-like structure of large n mixed correlation, which was found in [20].

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Appendix A. Proof of theorem 3.1

Proof. Note that $\hat{\psi}_n(y)$ is piecewise analytical in the connected domains separated by $\tilde{\gamma}^{(i)}$'s. In each such domain, it behaves at ∞ like

$$\hat{\psi}_n(y) \sim_\infty \sqrt{h_n}\, y^{-n-1}\, \mathrm{e}^{V_2(y)}(1 + O(1/y)). \tag{A.1}$$

Let $f_n(x, y)$ be the primitive of $\hat{\psi}_n(y)\,\mathrm{e}^{xy}$ which vanishes at ∞ in sectors where $\mathrm{Re} V_2 < 0$:

$$f_n(x, y) = \int^y \hat{\psi}_n(y'')\,\mathrm{e}^{xy''}, \tag{A.2}$$

$f_n(x, y)$ behaves at ∞ in each connected domain like

$$f_n(x, y) \sim_\infty \frac{1}{\tilde{t}}\sqrt{h_n}\, y^{-n-d_2-1}\,\mathrm{e}^{V_2(y)+xy}\left(1 + \sum_{k=1}^{\infty} f_{n,k}(x) y^{-k}\right) \tag{A.3}$$

where each $f_{n,k}(x)$ is a polynomial in x.

By definition of $L_{nm}(x)$ we have

$$L_{nm}(x) = \frac{1}{2\mathrm{i}\pi}\sum_{i=1}^{d_2}\int_{\tilde{\gamma}^{(i)}} \mathrm{d}y\,\phi_m(y)\,\mathrm{e}^{-xy}(f_n(x, y_+) - f_n(x, y_-))$$

$$= \frac{1}{2\mathrm{i}\pi}\sum_{\text{domains } D}\int_{\partial D} \mathrm{d}y\,\phi_m(y)\,\mathrm{e}^{-xy} f_n(x, y)$$

$$= \frac{1}{2\mathrm{i}\pi} \sum_{\text{domains } D} \int_{\partial D} \mathrm{d}y q_m(y) y^{-n-d_2-1}\,\mathrm{e}^{-xy} \left(1 + \sum_{k=1}^{\infty} f_{n,k}(x) y^{-k}\right)$$

$$= \operatorname*{Res}_{\infty} \mathrm{d}y\, q_m(y) y^{-n-d_2-1}\,\mathrm{e}^{-xy} \left(1 + \sum_{k=1}^{\infty} f_{n,k}(x) y^{-k}\right). \tag{A.4}$$

This expression vanishes if $m < n + d_2$ and is clearly a polynomial in x.

Now, compute

$$\sum_{i=1}^{d_2} \psi_n^{(i)}(x)\hat{\phi}_m^{(i)}(x) = \frac{1}{2\mathrm{i}\pi\sqrt{h_n h_m}} \sum_{i=1}^{d_2} \int_{\tilde{\gamma}^{(i)}} \mathrm{d}y \int_{\tilde{\tilde{\gamma}}^{(i)}} \mathrm{d}y'' \int_{\Gamma} \mathrm{d}x'\mathrm{d}y'\,\mathrm{e}^{-V_2(y)}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-V_1(x')}$$
$$\times \frac{1}{x-x'} \frac{V_2'(y'') - V_2'(y')}{y''-y'} p_n(x') q_m(y)\,\mathrm{e}^{xy''}\,\mathrm{e}^{-xy}\,\mathrm{e}^{-x'y'}$$
$$= \sum_{i=1}^{d_2} \sum_{j,k} \frac{\kappa_{kj}}{2\mathrm{i}\pi\sqrt{h_n h_m}} \int_{\tilde{\gamma}^{(i)}} \mathrm{d}y \int_{\tilde{\tilde{\gamma}}^{(i)}} \mathrm{d}y'' \int_{\gamma^{(k)}} \mathrm{d}x' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y'\,\mathrm{e}^{-V_2(y)}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}$$
$$\times\,\mathrm{e}^{-V_1(x')} \frac{1}{x-x'} \frac{V_2'(y'') - V_2'(y')}{y''-y'} p_n(x') q_m(y)\,\mathrm{e}^{xy''}\,\mathrm{e}^{-xy}\,\mathrm{e}^{-x'y'}$$
$$= \sum_{i=1}^{d_2} \sum_{j,k} \frac{\kappa_{kj}}{2\mathrm{i}\pi\sqrt{h_n h_m}} \int_{\tilde{\gamma}^{(i)}} \mathrm{d}y \int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} \mathrm{d}y'' \int_{\gamma^{(k)}} \mathrm{d}x' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y'\,\mathrm{e}^{-V_2(y)}$$
$$\times\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-V_1(x')} \frac{1}{x-x'} \frac{V_2'(y'') - V_2'(y')}{y''-y'} p_n(x') q_m(y)\,\mathrm{e}^{xy''}\,\mathrm{e}^{-xy}\,\mathrm{e}^{-}x'y'. \tag{A.5}$$

Moreover, we have

$$= \frac{1}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} \mathrm{d}y'' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y'\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')} \frac{V_2'(y'') - V_2'(y')}{y''-y'}\,\mathrm{e}^{xy''}\,\mathrm{e}^{-x'y'}$$
$$= \frac{1}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} \mathrm{d}y'' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y' \frac{V_2'(y'')}{y''-y'}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-x'y'}\,\mathrm{e}^{xy''}$$
$$- \frac{1}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} \mathrm{d}y'' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y' \frac{V_2'(y')}{y''-y'}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-x'y'}\,\mathrm{e}^{xy''}$$
$$= \frac{1}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} \mathrm{d}y'' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y' \frac{1}{(y''-y')^2}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-x'y'}\,\mathrm{e}^{xy''}$$
$$- \frac{1}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} \mathrm{d}y'' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y' \frac{x}{y''-y'}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-x'y'}\,\mathrm{e}^{xy''}$$
$$- \frac{\delta_{ij}}{2\mathrm{i}\pi} \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y' \left[\frac{1}{y''-y'}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-x'y'}\,\mathrm{e}^{xy''}\right]$$
$$- \frac{1}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} \mathrm{d}y'' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y' \frac{1}{(y''-y')^2}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-x'y'}\,\mathrm{e}^{xy''}$$
$$+ \frac{1}{2\mathrm{i}\pi} \int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} \mathrm{d}y'' \int_{\tilde{\gamma}^{(j)}} \mathrm{d}y' \frac{x'}{y''-y'}\,\mathrm{e}^{V_2(y'')}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-x'y'}\,\mathrm{e}^{xy''}$$
$$= \delta_{ij} \operatorname*{Res}_{y} \mathrm{d}y' \frac{1}{y-y'}\,\mathrm{e}^{V_2(y)}\,\mathrm{e}^{-V_2(y')}\,\mathrm{e}^{-x'y'}\,\mathrm{e}^{xy}$$

$$
\begin{aligned}
&\quad -\frac{1}{2i\pi}\int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} dy'' \int_{\tilde{\gamma}^{(j)}} dy' \frac{x-x'}{y''-y'}\, e^{V_2(y'')}\, e^{-V_2(y')}\, e^{-x'y'}\, e^{xy''}\\
&= -\delta_{ij}\, e^{(x-x')y}\\
&\quad -\frac{1}{2i\pi}\int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} dy'' \int_{\tilde{\gamma}^{(j)}} dy' \frac{x-x'}{y''-y'}\, e^{V_2(y'')}\, e^{-V_2(y')}\, e^{-x'y'}\, e^{x}y''.
\end{aligned} \tag{A.6}
$$

Therefore,

$$
\begin{aligned}
&\sum_{i=1}^{d_2}\psi_n^{(i)}(x)\hat{\phi}_m^{(i)}(x)\\
&= -\sum_{i=1}^{d_2}\sum_{j,k}\frac{\kappa_{kj}\delta_{ij}}{\sqrt{h_nh_m}}\int_{\tilde{\gamma}^{(i)}} dy\int_{\gamma^{(k)}} dx'\, e^{-V_2(y)}\, e^{-V_1(x')}\frac{1}{x-x'}p_n(x')q_m(y)\, e^{-x'y}\\
&\quad -\sum_{i=1}^{d_2}\sum_{j,k}\frac{\kappa_{kj}}{2i\pi\sqrt{h_nh_m}}\int_{\tilde{\gamma}^{(i)}} dy\int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} dy''\int_{\gamma^{(k)}} dx'\int_{\tilde{\gamma}^{(j)}} dy'\\
&e^{-V_2(y)}\, e^{V_2(y'')}\, e^{-V_2(y')}\, e^{-V_1(x')}\frac{1}{y''-y'}p_n(x')q_m(y)\, e^{xy''}\, e^{-xy}\, e^{-x'y'} = -\left(\frac{1}{x-Q}\right)_{nm}\\
&\quad -\sum_{i=1}^{d_2}\sum_{j,k}\frac{\kappa_{kj}}{2i\pi\sqrt{h_nh_m}}\int_{\tilde{\gamma}^{(i)}} dy\int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} dy''\int_{\gamma^{(k)}} dx'\int_{\tilde{\gamma}^{(j)}} dy'\\
&e^{-V_2(y)}\, e^{V_2(y'')}\, e^{-V_2(y')}\, e^{-V_1(x')}\frac{1}{y''-y'}p_n(x')q_m(y)\, e^{xy''}\, e^{-xy}\, e^{-x'y'}\\
&= -\left(\frac{1}{x-Q}\right)_{nm} - \sum_{i=1}^{d_2}\frac{1}{2i\pi}\int_{\tilde{\gamma}^{(i)}} dy\int_{\tilde{\tilde{\gamma}}_+^{(i)}(y)\cup\tilde{\tilde{\gamma}}_-^{(i)}(y)} dy''\hat{\psi}_n(y'')\phi_m(y)\, e^{xy''}\, e^{-xy}\\
&= -\left(\frac{1}{x-Q}\right)_{nm} + L_{nm}(x).
\end{aligned} \tag{A.7}
$$

Using equation (2.52), this also proves that L is a left inverse of $x - Q$. □

Appendix B. Proof of determinantal formulae

B.1. Inverses

1.

$$
\begin{aligned}
&\left(\Pi_{n-1}\left(1-\frac{\psi_\infty \mathbf{e}_n^t}{\psi_n}\right)\Pi_n R\Pi_{n-1}\right)\cdot(\Pi_{n-1}(x-Q)\Pi_{n-1})\\
&\qquad = \Pi_{n-1}\left(1-\frac{\psi_\infty \mathbf{e}_n^t}{\psi_n}\right)\Pi_n R(x-Q)\Pi_{n-1}\\
&\qquad = \Pi_{n-1}\left(1-\frac{\psi_\infty \mathbf{e}_n^t}{\psi_n}\right)\Pi_n\left(1-\frac{\psi_\infty \mathbf{e}_0^t}{\psi_0}\right)\Pi_{n-1}\\
&\qquad = \Pi_{n-1}\left(1-\frac{\psi_\infty \mathbf{e}_n^t}{\psi_n}-\frac{\psi_\infty \mathbf{e}_0^t}{\psi_0}+\frac{\psi_\infty \mathbf{e}_n^t}{\psi_n}\frac{\psi_\infty \mathbf{e}_0^t}{\psi_0}\right)\Pi_{n-1}\\
&\qquad = \Pi_{n-1}\left(1-\frac{\psi_\infty \mathbf{e}_n^t}{\psi_n}\right)\Pi_{n-1}\\
&\qquad = \mathrm{Id}_n.
\end{aligned} \tag{B.1}
$$

2. Inverse of $\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}$:

$$\left(\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\right).\left(\Pi_{n-1}\left(1+\frac{\mathbf{e}_{n-1}\hat{\phi}_\infty^t}{\hat{\phi}_{n-1}}\Pi^n\right)(x-Q)\Pi_{n-1}\right)$$
$$= \Pi_{n-1}\big(R+\psi_\infty\hat{\phi}_\infty^t\Pi_{n-1}\big)\left(1+\frac{\mathbf{e}_{n-1}\hat{\phi}_\infty^t}{\hat{\phi}_{n-1}}\Pi^n\right)(x-Q)\Pi_{n-1}$$
$$= \Pi_{n-1}\left(R+R\,\frac{\mathbf{e}_{n-1}\hat{\phi}_\infty^t}{\hat{\phi}_{n-1}}\Pi^n+\psi_\infty\hat{\phi}_\infty^t\Pi_{n-1}+\psi_\infty\hat{\phi}_\infty^t\frac{\mathbf{e}_{n-1}\hat{\phi}_\infty^t}{\hat{\phi}_{n-1}}\Pi^n\right)$$
$$(x-Q)\Pi_{n-1} = \Pi_{n-1}\big(R+\psi_\infty\hat{\phi}_\infty^t\big(\Pi_{n-1}+\Pi^n\big)\big)(x-Q)\Pi_{n-1}$$
$$= \Pi_{n-1}\big(R+\psi_\infty\hat{\phi}_\infty^t\big)(x-Q)\Pi_{n-1}$$
$$= \mathrm{Id}_n. \tag{B.2}$$

3. Inverse of $\Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1}$:

$$(\Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1}).\left(\Pi_{n-1}\tilde{R}^t\Pi_n\left(1-\frac{1}{K_{n+1}}\psi_\infty\phi_\infty^t\right)\Pi_nR\Pi_{n-1}\right)$$
$$= \Pi_{n-1}(x-Q)(y-P^t)\tilde{R}^t\Pi_n\left(1-\frac{1}{K_{n+1}}\psi_\infty\phi_\infty^t\right)\Pi_nR\Pi_{n-1}$$
$$= \Pi_{n-1}(x-Q)\left(\Pi_n-\frac{\mathbf{e}_0\phi_\infty^t}{\phi_0}\Pi_n\right)\left(1-\frac{1}{K_{n+1}}\psi_\infty\phi_\infty^t\right)\Pi_nR\Pi_{n-1}$$
$$= \Pi_{n-1}(x-Q)\left(\Pi_n-\Pi_n\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}-\frac{\mathbf{e}_0\phi_\infty^t}{\phi_0}\Pi_n\right.$$
$$\left.+\frac{\mathbf{e}_0\phi_\infty^t}{\phi_0}\Pi_n\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)\Pi_nR\Pi_{n-1}$$
$$= \Pi_{n-1}(x-Q)\left(\Pi_n-\Pi_n\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}-\frac{\mathbf{e}_0\phi_\infty^t}{\phi_0}\Pi_n+\frac{\mathbf{e}_0\phi_\infty^t}{\phi_0}\right)\Pi_nR\Pi_{n-1}$$
$$= \Pi_{n-1}(x-Q)\left(\Pi_n-\Pi_n\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\Pi_n\right)R\Pi_{n-1}$$
$$= \Pi_{n-1}(x-Q)\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\Pi_n\right)R\Pi_{n-1}$$
$$= \Pi_{n-1}(x-Q)R\Pi_{n-1}$$
$$= \mathrm{Id}_n. \tag{B.3}$$

4. Proof that

$$\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1} = \Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\Pi_{n-1}+J_n\Pi_{n-1}\psi_\infty\phi_\infty^t\Pi_{n-1}$$
$$= \Pi_{n-1}\frac{1}{x-Q}\Pi_{n-2}\frac{1}{y-P^t}\Pi_{n-1}+J_{n-1}\Pi_{n-1}\psi_\infty\phi_\infty^t\Pi_{n-1}. \tag{B.4}$$

We have

$$\left(\frac{1}{x-Q}\frac{1}{y-P^t}\right)_{nm} = \frac{1}{\sqrt{h_nh_m}}\int_\Gamma q_m(y')\frac{1}{y-y'}\frac{1}{x-x'}p_n(x')\omega(x',y')\,\mathrm{d}x'\,\mathrm{d}y'$$
$$= \frac{1}{\sqrt{h_nh_m}}\int_\Gamma \frac{q_m(y)-(q_m(y)-q_m(y'))}{y-y'}\,\frac{p_n(x)-(p_n(x)-p_n(x'))}{x-x'}\,\mathrm{d}\mu(x',y')$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{h_n h_m}} \int_\Gamma \frac{q_m(y)}{y-y'} \frac{p_n(x)}{x-x'} \omega(x',y')\,\mathrm{d}x'\mathrm{d}y' \\
&\quad - \frac{1}{\sqrt{h_n h_m}} \int_\Gamma \frac{q_m(y)-q_m(y')}{y-y'} \frac{p_n(x)}{x-x'} \omega(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad - \frac{1}{\sqrt{h_n h_m}} \int_\Gamma \frac{q_m(y)}{y-y'} \frac{p_n(x)-p_n(x')}{x-x'} \omega(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad + \frac{1}{\sqrt{h_n h_m}} \int_\Gamma \frac{q_m(y)-q_m(y')}{y-y'} \frac{p_n(x)-p_n(x')}{x-x'} \omega(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&= \psi_n(x)\phi_m(y) J_0(x,y) \\
&\quad + \sum_{l=0}^{m-1} \frac{1}{\sqrt{h_n h_m}} \int_\Gamma \tilde{R}_{ml}(y) q_l(y') \frac{p_n(x)}{x-x'} \omega(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad + \sum_{j=0}^{n-1} \frac{1}{\sqrt{h_n h_m}} \int_\Gamma \frac{q_m(y)}{y-y'} R_{nj}(x) p_j(x') \omega(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&\quad + \sum_{j=0}^{n-1}\sum_{l=0}^{m-1} \frac{1}{\sqrt{h_n h_m}} \int_\Gamma \tilde{R}_{ml}(y) R_{nj}(x) q_l(y') p_j(x') \omega(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\
&= \big(J_0(x,y)\psi_\infty(x)\phi^t_\infty(y) + \psi_\infty(x)\hat{\phi}^t_\infty(x)\tilde{R}^t(y) \\
&\quad + R(x)\hat{\psi}_\infty(y)\phi^t_\infty(y) + R(x)\tilde{R}^t(y)\big)_{nm}.
\end{aligned} \tag{B.5}
$$

Thus,

$$
\frac{1}{x-Q}\frac{1}{y-P^t} = J_0\psi_\infty\phi^t_\infty + \psi_\infty\hat{\phi}^t_\infty\tilde{R}^t + R\hat{\psi}_\infty\phi^t_\infty + R\tilde{R}^t. \tag{B.6}
$$

Now, we take the Π_{n-1} projection:

$$
\begin{aligned}
&\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1} - J_0\Pi_{n-1}\psi_\infty\phi^t_\infty\Pi_{n-1} \\
&\qquad = \Pi_{n-1}\psi_\infty\hat{\phi}^t_\infty\tilde{R}^t\Pi_{n-1} + \Pi_{n-1}R\hat{\psi}_\infty\phi^t_\infty\Pi_{n-1} + \Pi_{n-1}R\tilde{R}^t\Pi_{n-1} \\
&\qquad = \Pi_{n-1}\psi_\infty\hat{\phi}^t_\infty\Pi_{n-1}\tilde{R}^t\Pi_{n-1} + \Pi_{n-1}R\Pi_{n-1}\hat{\psi}_\infty\phi^t_\infty\Pi_{n-1} \\
&\qquad\quad + \Pi_{n-1}R\Pi_{n-1}\tilde{R}^t\Pi_{n-1} \\
&\qquad = \Pi_{n-1}\big(R+\psi_\infty\hat{\phi}^t_\infty\big)\Pi_{n-1}\big(\tilde{R}^t+\hat{\psi}_\infty\phi^t_\infty\big)\Pi_{n-1} \\
&\qquad\quad - \Pi_{n-1}\psi_\infty\hat{\phi}^t_\infty\Pi_{n-1}\hat{\psi}_\infty\phi^t_\infty\Pi_{n-1} \\
&\qquad = (J_n-J_0)\Pi_{n-1}\psi_\infty\phi^t_\infty\Pi_{n-1} + \Pi_{n-1}\big(R+\psi_\infty\hat{\phi}^t_\infty\big)\Pi_{n-1}\big(\tilde{R}^t+\hat{\psi}_\infty\phi^t_\infty\big)\Pi_{n-1} \\
&\qquad = \Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\Pi_{n-1} + J_n\Pi_{n-1}\psi_\infty\phi^t_\infty\Pi_{n-1}.
\end{aligned} \tag{B.7}
$$

Similarly, we have

$$
\begin{aligned}
&\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1} - J_0\Pi_{n-1}\psi_\infty\phi^t_\infty\Pi_{n-1} \\
&\qquad = \Pi_{n-1}\psi_\infty\hat{\phi}^t_\infty\tilde{R}^t\Pi_{n-1} + \Pi_{n-1}R\hat{\psi}_\infty\phi^t_\infty\Pi_{n-1} + \Pi_{n-1}R\tilde{R}^t\Pi_{n-1} \\
&\qquad = \Pi_{n-1}\psi_\infty\hat{\phi}^t_\infty\Pi_{n-2}\tilde{R}^t\Pi_{n-1} + \Pi_{n-1}R\Pi_{n-2}\hat{\psi}_\infty\phi^t_\infty\Pi_{n-1} \\
&\qquad\quad + \Pi_{n-1}R\Pi_{n-2}\tilde{R}^t\Pi_{n-1} \\
&\qquad = \Pi_{n-1}\big(R+\psi_\infty\hat{\phi}^t_\infty\big)\Pi_{n-2}\big(\tilde{R}^t+\hat{\psi}_\infty\phi^t_\infty\big)\Pi_{n-1} \\
&\qquad\quad - \Pi_{n-1}\psi_\infty\hat{\phi}^t_\infty\Pi_{n-2}\hat{\psi}_\infty\phi^t_\infty\Pi_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&= (J_{n-1} - J_0)\Pi_{n-1}\psi_\infty\phi_\infty^t\Pi_{n-1} \\
&\quad + \Pi_{n-1}\big(R + \psi_\infty\hat{\phi}_\infty^t\big)\Pi_{n-2}\big(\tilde{R}^t + \hat{\psi}_\infty\phi_\infty^t\big)\Pi_{n-1} \\
&= \Pi_{n-1}\frac{1}{x-Q}\Pi_{n-2}\frac{1}{y-P^t}\Pi_{n-1} + J_{n-1}\Pi_{n-1}\psi_\infty\phi_\infty^t\Pi_{n-1}.
\end{aligned}
\tag{B.8}
$$

5. Inverse of $\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1}$:
First, note that

$$
\begin{aligned}
\hat{\phi}_\infty^t\Pi^{n-1}(x-Q)\Pi_{n-1} &= \hat{\phi}_\infty^t(1-\Pi_{n-2})(x-Q)\Pi_{n-1} \\
&= \hat{\phi}_\infty^t(x-Q)\Pi_{n-1} - \hat{\phi}_\infty^t\Pi_{n-2}(x-Q)\Pi_{n-1} \\
&= \frac{\mathbf{e}_0^t}{\psi_0} - \hat{\phi}_\infty^t\Pi_{n-2}(x-Q).
\end{aligned}
\tag{B.9}
$$

Now, compute

$$
\begin{aligned}
&\Pi_{n-1}(y-P^t)\left(\Pi_{n-2} + \Pi^{n-1}\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_{n-1}}\Pi^{n-1}\right)(x-Q)\Pi_{n-1} \\
&\quad = \Pi_{n-1}(y-P^t)\left(\Pi_{n-2}(x-Q) + \frac{1}{J_{n-1}}\Pi^{n-1}\hat{\psi}_\infty\left(\frac{\mathbf{e}_0^t}{\psi_0} - \hat{\phi}_\infty^t\Pi_{n-2}(x-Q)\right)\right) \\
&\quad = \Pi_{n-1}(y-P^t)\left(\left(1 - \Pi^{n-1}\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_{n-1}}\right)\Pi_{n-2}(x-Q) + \Pi^{n-1}\frac{\hat{\psi}_\infty\,\mathbf{e}_0^t}{J_{n-1}\psi_0}\right).
\end{aligned}
\tag{B.10}
$$

Thus,

$$
\begin{aligned}
&\Pi_{n-1}(y-P^t)\left(\Pi_{n-2} + \Pi^{n-1}\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_{n-1}}\Pi^{n-1}\right)(x-Q)\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1} \\
&\quad = \Pi_{n-1}(y-P^t)\left(1 - \Pi^{n-1}\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_{n-1}}\right)\Pi_{n-2}\frac{1}{y-P^t}\Pi_{n-1} \\
&\qquad + \Pi_{n-1}(y-P^t)\Pi^{n-1}\frac{\hat{\psi}_\infty}{J_{n-1}\psi_0}\,\mathbf{e}_0^t\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1} \\
&\quad = \Pi_{n-1}(y-P^t)\left(1 - \Pi^{n-1}\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_{n-1}}\right)\Pi_{n-2}\big(\tilde{R}^t + \hat{\psi}_\infty\phi_\infty^t\big)\Pi_{n-1} \\
&\qquad + \Pi_{n-1}(y-P^t)\Pi^{n-1}\frac{\hat{\psi}_\infty}{J_{n-1}\psi_0}\,\mathbf{e}_0^t\big(R + \psi_\infty\hat{\phi}_\infty^t\big)\Pi_{n-2}\big(\tilde{R}^t + \hat{\psi}_\infty\phi_\infty^t\big)\Pi_{n-1} \\
&\qquad + \Pi_{n-1}(y-P^t)\Pi^{n-1}\frac{\hat{\psi}_\infty}{J_{n-1}\psi_0}\,\mathbf{e}_0^t\big(J_{n-1}\psi_\infty\phi_\infty^t\big)\Pi_{n-1} \\
&\quad = \Pi_{n-1}(y-P^t)\left(1 - \Pi^{n-1}\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_{n-1}}\right)\big(\tilde{R}^t + \Pi_{n-2}\hat{\psi}_\infty\phi_\infty^t\big)\Pi_{n-1} \\
&\qquad + \Pi_{n-1}(y-P^t)\Pi^{n-1}\frac{\hat{\psi}_\infty}{J_{n-1}}\hat{\phi}_\infty^t\big(\tilde{R}^t + \Pi_{n-2}\hat{\psi}_\infty\phi_\infty^t\big)\Pi_{n-1} \\
&\qquad + \Pi_{n-1}(y-P^t)\Pi^{n-1}\hat{\psi}_\infty\phi_\infty^t\Pi_{n-1} \\
&\quad = \Pi_{n-1}(y-P^t)\big(\tilde{R}^t + \Pi_{n-2}\hat{\psi}_\infty\phi_\infty^t\big)\Pi_{n-1} + \Pi_{n-1}(y-P^t)\Pi^{n-1}\hat{\psi}_\infty\phi_\infty^t\Pi_{n-1} \\
&\quad = \Pi_{n-1}(y-P^t)\big(\tilde{R}^t + \hat{\psi}_\infty\phi_\infty^t\big)\Pi_{n-1} \\
&\quad = \mathrm{Id}_n.
\end{aligned}
\tag{B.11}
$$

B.2. Determinants

1. Determinant of $\Pi_{n-1}(x-Q)\Pi_{n-1}$. This is a classical result and can be found in the literature.
   One possible proof is that $\det(\Pi_{n-1}(x-Q)\Pi_{n-1})$ is a monic polynomial of degree n. Consider x as a zero of $p_n(x)$, this implies that $\Pi_{n-1}\psi_\infty(x) = \Pi_n\psi_\infty(x)$, thus
   $$\Pi_{n-1}(x-Q)\Pi_{n-1}\psi_\infty(x) = \Pi_{n-1}(x-Q)\Pi_n\psi_\infty(x) = \Pi_{n-1}(x-Q)\psi_\infty(x) = 0. \tag{B.12}$$
   Thus, the n zeros of ψ_n are also the zeros of $\det(\Pi_{n-1}(x-Q)\Pi_{n-1})$. The converse is easy too.
   Thus, we have
   $$\det(\Pi_{n-1}(x-Q)\Pi_{n-1}) = p_n(x), \qquad \det(\Pi_{n-1}(y-P)\Pi_{n-1}) = q_n(y). \tag{B.13}$$
2. Determinant of $\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}$:
   Compute
   $$\begin{aligned} p_n(x)\det\left(\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\right) &= \det\left(\Pi_{n-1}(x-Q)\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\right)\\ &= \det\left(\Pi_{n-1}(x-Q)\Pi_{n-1}\left(R+\psi_\infty\hat{\phi}^t_\infty\right)\Pi_{n-1}\right)\\ &= \det\left(\Pi_{n-1}(x-Q)\left(\Pi_n - \mathbf{e}_n\,\mathbf{e}_n^t\right)\left(R+\psi_\infty\hat{\phi}^t_\infty\right)\Pi_{n-1}\right)\\ &= \det\left(\Pi_{n-1}(x-Q)\left(R+\psi_\infty\hat{\phi}^t_\infty\right)\Pi_{n-1}\right.\\ &\qquad \left.-\Pi_{n-1}(x-Q)\,\mathbf{e}_n\,\mathbf{e}_n^t\left(R+\psi_\infty\hat{\phi}^t_\infty\right)\Pi_{n-1}\right)\\ &= \det\left(\mathrm{Id}_n + \gamma_n\,\mathbf{e}_{n-1}\,\mathbf{e}_n^t\left(R+\psi_\infty\hat{\phi}^t_\infty\right)\Pi_{n-1}\right)\\ &= 1+\gamma_n\,\mathbf{e}_n^t\left(R+\psi_\infty\hat{\phi}^t_\infty\right)\mathbf{e}_{n-1}\\ &= 1+\gamma_n\left(R_{n,n-1}+\psi_n\hat{\phi}_{n-1}\right)\\ &= 1+\gamma_n\left(-\frac{1}{\gamma_n}+\psi_n\hat{\phi}_{n-1}\right)\\ &= \gamma_n\psi_n\hat{\phi}_{n-1} \end{aligned} \tag{B.14}$$
   i.e.
   $$\det\left(\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\right) = \frac{\gamma_n\,\mathrm{e}^{-V_1}}{\sqrt{h_n}}\hat{\phi}_{n-1} = \frac{\mathrm{e}^{-V_1}}{\sqrt{h_{n-1}}}\hat{\phi}_{n-1}. \tag{B.15}$$
3. Kernel K_n:
   $$\begin{aligned} &\det(\Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1})\\ &\quad= \det(\Pi_{n-1}(x-Q)\Pi_{n-1}(y-P^t)\Pi_{n-1} + \Pi_{n-1}(x-Q)\Pi^n(y-P^t)\Pi_{n-1})\\ &\quad= \det\left(\Pi_{n-1}(x-Q)\Pi_{n-1}(y-P^t)\Pi_{n-1} + \gamma_n^2\,\mathbf{e}_{n-1}\,\mathbf{e}_{n-1}^t\right)\\ &\quad= p_n(x)q_n(y)\det\left(\mathrm{Id}_n\right.\\ &\qquad\left.+\gamma_n^2\Pi_{n-1}\left(1-\frac{\psi_\infty\mathbf{e}_n^t}{\psi_n}\right)\Pi_n R\mathbf{e}_{n-1}\mathbf{e}_{n-1}^t\tilde{R}^t\Pi_n\left(1-\frac{\mathbf{e}_n\phi^t_\infty}{\phi_n}\right)\Pi_{n-1}\right)\\ &\quad= p_n(x)q_n(y)\det\left(\mathrm{Id}_n + \Pi_{n-1}\left(1-\frac{\psi_\infty\mathbf{e}_n^t}{\psi_n}\right)\mathbf{e}_n\mathbf{e}_n^t\left(1-\frac{\mathbf{e}_n\phi^t_\infty}{\phi_n}\right)\Pi_{n-1}\right)\\ &\quad= p_n(x)q_n(y)\left(1+\mathbf{e}_n^t\left(1-\frac{\mathbf{e}_n\phi^t_\infty}{\phi_n}\right)\Pi_{n-1}\left(1-\frac{\psi_\infty\,\mathbf{e}_n^t}{\psi_n}\right)\mathbf{e}_n\right) \end{aligned}$$

$$
\begin{aligned}
&= p_n(x)q_n(y)\left(1+\frac{K_n}{\psi_n\phi_n}\right)\\
&= p_n(x)q_n(y)\frac{K_{n+1}}{\psi_n\phi_n}\\
&= h_n\,\mathrm{e}^{V_1(x)+V_2(y)}K_{n+1}.
\end{aligned}
\tag{B.16}
$$

4. Kernel J_n:

$$
\begin{aligned}
\det&\left(\Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1}\right)\\
&= \det\left(J_n\Pi_{n-1}\psi_\infty\phi_\infty^t\Pi_{n-1}+\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\Pi_{n-1}\right)\\
&= \det\left(\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\Pi_{n-1}\right)\\
&\quad\times\left(1+J_n\phi_\infty^t\left(\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\Pi_{n-1}\right)^{-1}\psi_\infty\right)\\
&= \frac{\mathrm{e}^{-V_1-V_2}}{h_{n-1}}\hat\psi_{n-1}\hat\phi_{n-1}\\
&\quad\times\left(1+J_n\phi_\infty^t\left(\Pi_{n-1}\frac{1}{y-P^t}\Pi_{n-1}\right)^{-1}\left(\Pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\right)^{-1}\psi_\infty\right)\\
&= \frac{\mathrm{e}^{-V_1-V_2}}{h_{n-1}}\hat\psi_{n-1}\hat\phi_{n-1}\Bigg(1+J_n\phi_\infty^t\Pi_{n-1}(y-P^t)\\
&\quad\times\left(1+\Pi^n\frac{\hat\psi_\infty\,\mathbf{e}_{n-1}^t}{\hat\psi_{n-1}}\right)\Pi_{n-1}\left(1+\frac{\mathbf{e}_{n-1}\hat\phi_\infty^t}{\hat\phi_{n-1}}\Pi^n\right)(x-Q)\Pi_{n-1}\psi_\infty\Bigg)\\
&= \frac{\mathrm{e}^{-V_1-V_2}}{h_{n-1}}\hat\psi_{n-1}\hat\phi_{n-1}\Bigg(1+J_n\phi_\infty^t\Pi_{n-1}(y-P^t)\\
&\quad\times\left(1+\Pi^n\frac{\hat\psi_\infty\mathbf{e}_{n-1}^t}{\hat\psi_{n-1}}\right)\Pi_{n-1}\left(1+\frac{\mathbf{e}_{n-1}\hat\phi_\infty^t}{\hat\phi_{n-1}}\Pi^n\right)(x-Q)\Pi_{n-1}\psi_\infty\Bigg)\\
&= \frac{\mathrm{e}^{-V_1-V_2}}{h_{n-1}}\hat\psi_{n-1}\hat\phi_{n-1}\Bigg(1+J_n\phi_\infty^t\Pi_{n-1}(y-P^t)\\
&\quad\times\left(1+\Pi^n\frac{\hat\psi_\infty\mathbf{e}_{n-1}^t}{\hat\psi_{n-1}}\right)\left(\Pi_{n-2}+\mathbf{e}_{n-1}\,\mathbf{e}_{n-1}^t\right)\left(1+\frac{\mathbf{e}_{n-1}\hat\phi_\infty^t}{\hat\phi_{n-1}}\Pi^n\right)(x-Q)\Pi_{n-1}\psi_\infty\Bigg)\\
&= \frac{\mathrm{e}^{-V_1-V_2}}{h_{n-1}}\hat\psi_{n-1}\hat\phi_{n-1}\Bigg(1+J_n\phi_\infty^t\Pi_{n-1}(y-P^t)\\
&\quad\times\left(1+\Pi^n\frac{\hat\psi_\infty\mathbf{e}_{n-1}^t}{\hat\psi_{n-1}}\right)\mathbf{e}_{n-1}\,\mathbf{e}_{n-1}^t\left(1+\frac{\mathbf{e}_{n-1}\hat\phi_\infty^t}{\hat\phi_{n-1}}\Pi^n\right)(x-Q)\Pi_{n-1}\psi_\infty\Bigg)\\
&= \frac{\mathrm{e}^{-V_1-V_2}}{h_{n-1}}\hat\psi_{n-1}\hat\phi_{n-1}\Bigg(1+J_n\phi_\infty^t\Pi_{n-1}(y-P^t)\\
&\quad\times\left(\mathbf{e}_{n-1}+\Pi^n\frac{\hat\psi_\infty}{\hat\psi_{n-1}}\right)\left(\mathbf{e}_{n-1}^t+\frac{\hat\phi_\infty^t}{\hat\phi_{n-1}}\Pi^n\right)(x-Q)\Pi_{n-1}\psi_\infty\Bigg)
\end{aligned}
$$

$$= \frac{e^{-V_1-V_2}}{h_{n-1}}\hat{\psi}_{n-1}\hat{\phi}_{n-1}\Bigg(1 + J_n\phi^t_\infty\Pi_{n-1}(y - P^t)$$
$$\times\,\Pi^{n-1}\frac{\hat{\psi}_\infty}{\hat{\psi}_{n-1}}\frac{\hat{\phi}^t_\infty}{\hat{\phi}_{n-1}}\Pi^{n-1}(x - Q)\Pi_{n-1}\psi_\infty\Bigg)$$
$$= \frac{e^{-V_1-V_2}}{h_{n-1}}\hat{\psi}_{n-1}\hat{\phi}_{n-1}\left(1 + J_n\phi^t_\infty\Pi_{n-1}(y - P^t)\frac{\hat{\psi}_\infty}{\hat{\psi}_{n-1}}\frac{\hat{\phi}^t_\infty}{\hat{\phi}_{n-1}}(x - Q)\Pi_{n-1}\psi_\infty\right)$$
$$= \frac{e^{-V_1-V_2}}{h_{n-1}}\hat{\psi}_{n-1}\hat{\phi}_{n-1}\left(1 + J_n\phi^t_\infty\frac{\mathbf{e}_0}{\hat{\psi}_{n-1}\phi_0}\frac{\mathbf{e}^t_0}{\hat{\phi}_{n-1}\psi_0}\psi_\infty\right)$$
$$= \frac{e^{-V_1-V_2}}{h_{n-1}}\hat{\psi}_{n-1}\hat{\phi}_{n-1}\left(1 + \frac{J_n}{\hat{\psi}_{n-1}\hat{\phi}_{n-1}}\right)$$
$$= \frac{e^{-V_1-V_2}}{h_{n-1}}J_{n-1}. \tag{B.17}$$

Appendix C. Matrices U and $\tilde{U}$

C.1. Definitions

The following matrices play an important role.

Define

$$U_n(x, y) := \Pi_n(y - P^t)R(x)\Pi^{n-1}, \qquad \tilde{U}_n(x, y) := \Pi_n(x - Q^t)\tilde{R}(y)\Pi^{n-1}. \tag{C.1}$$

Note that since R is lower triangular and P is finite band, $U_n(x, y)$ (resp. $\tilde{U}_n(x, y)$) is a small lower triangular matrix of size $d_1 + 1$ (resp. $d_2 + 1$):

$$U_n(x, y) = \Pi^{n-d_1}_n(y - P^t)R(x)\Pi^{n-1}_{n+d_1-1}, \tag{C.2}$$

$U_n(x, y)$ (resp. $\tilde{U}_n(x, y)$) is linear in y (resp. x) and of degree at most d_1 in x (resp. d_2 in y).

$U_n(x, y)$ can also be rewritten as

$$U_n(x, y) = -\frac{y + V_1'(x)}{\gamma_n}\mathbf{e}_n\,\mathbf{e}^t_{n-1} - \Pi_n\frac{V_1'(x) - V_1'(Q)}{x - Q}\Pi^{n-1}. \tag{C.3}$$

Proof.

$$\begin{aligned}
U_n(x, y) &= \Pi^{n-d_1}_n(y - P^t)R(x)\Pi^{n-1}_{n+d_1-1}\\
&= \Pi^{n-d_1}_n(y + V_1'(x) - V_1'(x) + V_1'(Q) - V_1'(Q) - P^t)R(x)\Pi^{n-1}_{n+d_1-1}\\
&= \Pi^{n-d_1}_n(y + V_1'(x))R(x)\Pi^{n-1}_{n+d_1-1} - \Pi^{n-d_1}_n(V_1'(x) - V_1'(Q))R(x)\Pi^{n-1}_{n+d_1-1}\\
&\quad - \Pi^{n-d_1}_n(V_1'(Q) + P^t)R(x)\Pi^{n-1}_{n+d_1-1}\\
&= \Pi^{n-d_1}_n(y + V_1'(x))R(x)\Pi^{n-1}_{n+d_1-1} - \Pi^{n-d_1}_n(V_1'(x) - V_1'(Q))R(x)\Pi^{n-1}_{n+d_1-1}\\
&= (y + V_1'(x))\Pi^{n-d_1}_n R(x)\Pi^{n-1}_{n+d_1-1} - \Pi^{n-d_1}_n\frac{V_1'(x) - V_1'(Q)}{x - Q}(x - Q)R(x)\Pi^{n-1}_{n+d_1-1}\\
&= -\frac{y + V_1'(x)}{\gamma_n}\mathbf{e}_n\,\mathbf{e}^t_{n-1} - \Pi^{n-d_1}_n\frac{V_1'(x) - V_1'(Q)}{x - Q}\Pi^{n-1}_{n+d_1-1}\\
&= -\frac{y + V_1'(x)}{\gamma_n}\mathbf{e}_n\,\mathbf{e}^t_{n-1} - \Pi_n\frac{V_1'(x) - V_1'(Q)}{x - Q}\Pi^{n-1}.
\end{aligned} \tag{C.4}$$

□

Also note that if $n > d_2 d_1$, we have

$$U_n(x, y) := \Pi_n R(x)(y - P^t)\Pi^{n-1}. \tag{C.5}$$

C.2. Some properties

- Multiplication by the Christoffel–Darboux matrix:

$$-U_n(x, y)A_n = B^t_{n+1} + (y - P^t)\,\mathbf{e}_n\,\mathbf{e}^t_n - \Pi_n(y - P^t)R\Pi^n(x - Q) \tag{C.6}$$

$$-A_nU_n(x, y) = B^t_{n-1} + \mathbf{e}_{n-1}\,\mathbf{e}^t_{n-1}(y - P^t) - (x - Q)\Pi_{n-1}(y - P^t)R\Pi^{n-1} \tag{C.7}$$

and subsequently

$$\Phi^t_\infty U_n(x, y)A_n\Psi_\infty = -\Phi^t_\infty B^t_{n+1}\Psi_\infty \tag{C.8}$$

$$\hat{\Phi}^t_\infty A_nU_n(x, y)\hat{\Psi}_\infty = -\hat{\Phi}^t_\infty B^t_{n-1}\hat{\Psi}_\infty. \tag{C.9}$$

Proof

$$\begin{aligned}
-U_n(x, y)A_n &= \Pi_n(y - P^t)R(x - Q)\Pi^n - \Pi_n(y - P^t)R\Pi^n(x - Q)\\
&= \Pi_n(y - P^t)\Pi^n - \Pi_n(y - P^t)R\Pi^n(x - Q)\\
&= [\Pi_n, y - P^t]\Pi^n + (y - P^t)\,\mathbf{e}_n\,\mathbf{e}^t_n - \Pi_n(y - P^t)R\Pi^n(x - Q)\\
&= B^t_{n+1} + (y - P^t)\,\mathbf{e}_n\,\mathbf{e}^t_n - \Pi_n(y - P^t)R\Pi^n(x - Q).
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
-A_nU_n(x, y) &= \Pi_{n-1}(x - Q)(y - P^t)R\Pi^{n-1} - (x - Q)\Pi_{n-1}(y - P^t)R\Pi^{n-1}\\
&= \Pi_{n-1}(1 + (y - P^t)(x - Q))R\Pi^{n-1} - (x - Q)\Pi_{n-1}(y - P^t)R\Pi^{n-1}\\
&= \Pi_{n-1}(y - P^t)\Pi^{n-1} + \Pi_{n-1}R\Pi^{n-1} - (x - Q)\Pi_{n-1}(y - P^t)R\Pi^{n-1}\\
&= \Pi_{n-1}(y - P^t)\Pi^{n-1} - (x - Q)\Pi_{n-1}(y - P^t)R\Pi^{n-1}\\
&= \Pi_{n-1}[y - P^t, \Pi^{n-1}] + \Pi_{n-1}\Pi^{n-1}(y - P^t)\\
&\quad - (x - Q)\Pi_{n-1}(y - P^t)R\Pi^{n-1}\\
&= \Pi_{n-1}[P^t, \Pi_{n-2}] + \mathbf{e}_{n-1}\,\mathbf{e}^t_{n-1}(y - P^t) - (x - Q)\Pi_{n-1}(y - P^t)R\Pi^{n-1}\\
&= B^t_{n-1} + \mathbf{e}_{n-1}\,\mathbf{e}^t_{n-1}(y - P^t) - (x - Q)\Pi_{n-1}(y - P^t)R\Pi^{n-1}.
\end{aligned} \tag{C.11}$$

- Inverse:

$$\tilde{U}^{t-1}_n(x, y) = \Pi^{n-d_2}_n(y - P^t)L(x)\Pi^{n-1}_{n+d_2-1}. \tag{C.12}$$

□

Proof

$$\begin{aligned}
\tilde{U}^t_n(x, y)\Pi^{n-d_2}_n(y - P^t)L(x)\Pi^{n-1}_{n+d_2-1} &= \Pi^{n-1}\tilde{R}^t(y)(x - Q)\Pi^{n-d_2}_n(y - P^t)L(x)\Pi^{n-1}_{n+d_2-1}\\
&= \Pi^{n-1}\tilde{R}^t(y)\Pi^n(x - Q)\Pi_n(y - P^t)\Pi_{n-1}L(x)\Pi^{n-1}_{n+d_2-1}\\
&= \Pi^{n-1}\tilde{R}^t(y)\Pi^n(x - Q)(y - P^t)\Pi_{n-1}L(x)\Pi^{n-1}_{n+d_2-1}\\
&= \Pi^{n-1}\tilde{R}^t(y)\Pi^n(y - P^t)(x - Q)\Pi_{n-1}L(x)\Pi^{n-1}_{n+d_2-1}\\
&= \Pi^{n-1}\tilde{R}^t(y)(y - P^t)(x - Q)L(x)\Pi^{n-1}_{n+d_2-1}\\
&= \Pi^{n-1}_{n+d_2-1}.
\end{aligned} \tag{C.13}$$

□

- Multiplication of the inverse by the Christoffel–Darboux matrix:

$$-\tilde{U}_n^{t-1}(x,y)A_n = B^t_{n-d_2}\Pi^n + (y-P^t)\,\mathbf{e}_n\,\mathbf{e}_n^t - \Pi_n^{n-d_2}(y-P^t)L(x)\Pi^n(x-Q) \quad (C.14)$$

$$A_n\tilde{U}_n^{t-1}(x,y) = -\Pi^n B^t_{n+d_2} + \Pi^n_{n+d_2-1}(y-P^t) - (x-Q)\Pi^n(y-P^t)L(x)\Pi^{n-1}_{n+d_2-1}. \quad (C.15)$$

In particular,

$$\Phi^t_\infty\tilde{U}_n^{t-1}(x,y)A_n\Psi_\infty = -\Phi^t_\infty B^t_{n-d_2}\Pi_{n-1}\Psi_\infty \quad (C.16)$$

and

$$\hat{\Phi}^t_\infty A_n\tilde{U}_n^{t-1}(x,y)\hat{\vec{\psi}} = -\hat{\Phi}^t_\infty\Pi^n B^t_{n+d_2}\hat{\vec{\psi}}. \quad (C.17)$$

Proof

$$\begin{aligned}\tilde{U}_n^{t-1}(x,y)A_n &= \Pi_n^{n-d_2}(y-P^t)L(x)[x-Q,\Pi_{n-1}]\\ &= \Pi_n^{n-d_2}(y-P^t)L(x)(x-Q)\Pi_{n-1} - \Pi_n^{n-d_2}(y-P^t)L(x)\Pi_{n-1}(x-Q)\\ &= \Pi_n^{n-d_2}(y-P^t)\Pi_{n-1} + \gamma_{n-d_2}L_{n-d_2-1,n-1}\,\mathbf{e}_{n-d_2}\,\mathbf{e}^t_{n-1}(x-Q). \quad (C.18)\end{aligned}$$

$$\begin{aligned}-\tilde{U}_n^{t-1}(x,y)A_n &= \Pi_n^{n-d_2}(y-P^t)L(x)[x-Q,\Pi^n]\\ &= \Pi_n^{n-d_2}(y-P^t)L(x)(x-Q)\Pi^n - \Pi_n^{n-d_2}(y-P^t)L(x)\Pi^n(x-Q)\\ &= \Pi_n^{n-d_2}(y-P^t)\Pi^n - \Pi_n^{n-d_2}(y-P^t)L(x)\Pi^n(x-Q)\\ &= \left[\Pi_n^{n-d_2}, y-P^t\right]\Pi^n + (y-P^t)\Pi_n^{n-d_2}\Pi^n\\ &\quad - \Pi_n^{n-d_2}(y-P^t)L(x)\Pi^n(x-Q)\\ &= B^t_{n-d_2}\Pi^n + (y-P^t)\,\mathbf{e}_n\,\mathbf{e}^t_n - \Pi_n^{n-d_2}(y-P^t)L(x)\Pi^n(x-Q). \quad (C.19)\end{aligned}$$

Then,

$$\begin{aligned}A_n\tilde{U}_n^{t-1}(x,y) &= -[x-Q,\Pi^n](y-P^t)L(x)\Pi^{n-1}_{n+d_2-1}\\ &= \Pi^n(x-Q)(y-P^t)L(x)\Pi^{n-1}_{n+d_2-1} - (x-Q)\Pi^n(y-P^t)L(x)\Pi^{n-1}_{n+d_2-1}\\ &= \Pi^n(1+(y-P^t)(x-Q))L(x)\Pi^{n-1}_{n+d_2-1}\\ &\quad - (x-Q)\Pi^n(y-P^t)L(x)\Pi^{n-1}_{n+d_2-1}\\ &= \Pi^n(y-P^t)\Pi^{n-1}_{n+d_2-1} + \Pi^n L(x)\Pi^{n-1}_{n+d_2-1}\\ &\quad - (x-Q)\Pi^n(y-P^t)L(x)\Pi^{n-1}_{n+d_2-1}\\ &= \Pi^n(y-P^t)\Pi_{n+d_2-1} - (x-Q)\Pi^n(y-P^t)L(x)\Pi^{n-1}_{n+d_2-1}\\ &= \Pi^n\left[(y-P^t),\Pi_{n+d_2-1}\right] + \Pi^n_{n+d_2-1}(y-P^t)\\ &\quad - (x-Q)\Pi^n(y-P^t)L(x)\Pi^{n-1}_{n+d_2-1}\\ &= -\Pi^n B^t_{n+d_2} + \Pi^n_{n+d_2-1}(y-P^t) - (x-Q)\Pi^n(y-P^t)L(x)\Pi^{n-1}_{n+d_2-1}. \quad (C.20)\end{aligned}$$

□

Appendix D. Proof of theorem 7.1

Proof. Let us introduce the following $n \times n$ matrices:

$$\zeta_n := \Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1} \tag{D.1}$$

$$\xi_n := \Pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\Pi_{n-1} \tag{D.2}$$

$$\omega_n := 1_n + \Pi_{n-1}(y-P^t)\Pi_{n-2}(x-Q)\Pi_{n-1} \tag{D.3}$$

and, according to formulae of section 5.1,

$$\det\zeta_n = h_n K_{n+1}, \qquad \zeta_n^{-1} = \Pi_{n-1}\tilde{R}^t\Pi_n\left(1-\frac{1}{K_{n+1}}\psi_\infty\phi_\infty^t\right)\Pi_n R\Pi_{n-1} \tag{D.4}$$

$$\det\xi_n = \frac{J_{n-1}}{h_{n-1}} \tag{D.5}$$

$$\xi_n^{-1} = \Pi_{n-1}(y-P^t)\left(\Pi_{n-2}+\frac{1}{J_{n-1}}\Pi^{n-1}\hat{\psi}_\infty\hat{\phi}_\infty^t\Pi^{n-1}\right)(x-Q)\Pi_{n-1}. \tag{D.6}$$

We have

$$\begin{aligned}\omega_n &= 1_n + \Pi_{n-1}(y-P^t)\Pi_{n-2}(x-Q)\Pi_{n-1}\\ &= 1_n + \Pi_{n-1}(y-P^t)(x-Q)\Pi_{n-1} - \Pi_{n-1}(y-P^t)\Pi^{n-1}(x-Q)\Pi_{n-1}\\ &= \zeta_n - \Pi_{n-1}(y-P^t)\Pi^{n-1}(x-Q)\Pi_{n-1}\\ &= \zeta_n\left(1-\frac{1}{\zeta_n}\Pi_{n-1}(y-P^t)\Pi^{n-1}(x-Q)\Pi_{n-1}\right).\end{aligned} \tag{D.7}$$

Thus,

$$\begin{aligned}\det\omega_n &= \det\zeta_n\det\left(1-\frac{1}{\zeta_n}\Pi_{n-1}(y-P^t)\Pi^{n-1}(x-Q)\Pi_{n-1}\right)\\ &= h_nK_{n+1}\det\left(1-\Pi_{n-1}\tilde{R}^t\Pi_n\left(1-\frac{1}{K_{n+1}}\psi_\infty\phi_\infty^t\right)\Pi_n R\right.\\ &\qquad\left.\times(y-P^t)\Pi^{n-1}(x-Q)\Pi_{n-1}\right)\\ &= h_nK_{n+1}\det\left(1-\Pi_{n-1}\tilde{R}^t\Pi_n\left(1-\frac{1}{K_{n+1}}\psi_\infty\phi_\infty^t\right)U_n(x-Q)\Pi_{n-1}\right)\\ &= h_nK_{n+1}\det\left(1-\Pi_n\left(1-\frac{1}{K_{n+1}}\psi_\infty\phi_\infty^t\right)U_n\tilde{U}_n^t\right)\end{aligned} \tag{D.8}$$

and

$$\begin{aligned}\det\omega_n &= h_nK_{n+1}\det\left(1-U_n\tilde{U}_n^t+\frac{1}{K_{n+1}}\Pi_n\psi_\infty\phi_\infty^tU_n\tilde{U}_n^t\right)\\ &= h_nK_{n+1}\det\left(1-U_n\tilde{U}_n^t\right)\left(1+\frac{1}{K_{n+1}}\phi_\infty^tU_n\tilde{U}_n^t\frac{1}{1-U_n\tilde{U}_n^t}\psi_\infty\right)\\ &= h_n\det\left(1-U_n\tilde{U}_n^t\right)\left(\phi_\infty^t\frac{1}{1-U_n\tilde{U}_n^t}\psi_\infty\right).\end{aligned} \tag{D.9}$$

Inverse,

$$\begin{aligned}\omega_n^{-1} &= \left(1-\frac{1}{\zeta_n}\Pi_{n-1}(y-P^t)\Pi^{n-1}(x-Q)\Pi_{n-1}\right)^{-1}\zeta_n^{-1}\\ &= \left(1-\Pi_{n-1}\tilde{R}^t\Pi_n\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)U_n(x-Q)\Pi_{n-1}\right)^{-1}\zeta_n^{-1}\\ &= \left(1+\Pi_{n-1}\tilde{R}^t\Pi_n\frac{1}{1-\Pi_n\big(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\big)U_n\tilde{U}_n^t}\Pi_n\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)U_n(x-Q)\Pi_{n-1}\right)\zeta_n^{-1}\\ &= \zeta_n^{-1}+\Pi_{n-1}\tilde{R}^t\Pi_n\frac{1}{1-\Pi_n\big(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\big)U_n\tilde{U}_n^t}\Pi_n\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)U_n\\ &\quad\times\tilde{U}_n^t\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)\Pi_nR\Pi_{n-1}\\ &= \Pi_{n-1}\tilde{R}^t\Pi_n\frac{1}{1-\Pi_n\big(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\big)U_n\tilde{U}_n^t}\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)\Pi_nR\Pi_{n-1}.\end{aligned}\tag{D.10}$$

Now, the formula of [8] gives $W_n=\det(1+\xi_n)$, thus we compute

$$\begin{aligned}W_n=\det(1+\xi_n)&=\det\xi_n\det\left(1+\xi_n^{-1}\right)\\ &=\det\xi_n\det\left(1+\Pi_{n-1}(y-P^t)\left(\Pi_{n-2}+\frac{1}{J_{n-1}}\Pi^{n-1}\hat{\psi}_\infty\hat{\phi}_\infty^t\Pi^{n-1}\right)(x-Q)\Pi_{n-1}\right)\\ &=\det\xi_n\det\left(\omega_n+\frac{1}{J_{n-1}}\Pi_{n-1}(y-P^t)\Pi^{n-1}\hat{\psi}_\infty\hat{\phi}_\infty^t\Pi^{n-1}(x-Q)\Pi_{n-1}\right)\\ &=\det\xi_n\det\omega_n\left(1+\frac{1}{J_{n-1}}\hat{\phi}_\infty^t\Pi^{n-1}(x-Q)\Pi_{n-1}\omega_n^{-1}\Pi_{n-1}(y-P^t)\Pi^{n-1}\hat{\psi}_\infty\right).\end{aligned}\tag{D.11}$$

Thus,

$$\begin{aligned}h_{n-1}W_n&=\det\omega_n\big(J_{n-1}+\hat{\phi}_\infty^t\Pi^{n-1}(x-Q)\Pi_{n-1}\omega_n^{-1}\Pi_{n-1}(y-P^t)\Pi^{n-1}\hat{\psi}_\infty\big)\\ &=\det\omega_n\left(J_{n-1}+\hat{\phi}_\infty^t\tilde{U}_n^t\frac{1}{1-\Pi_n\big(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\big)U_n\tilde{U}_n^t}\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)U_n\hat{\psi}_\infty\right),\end{aligned}\tag{D.12}$$

i.e.

$$W_n=\gamma_n^2K_{n+1}J_{n-1}\det\left(1-\Pi_n\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)U_n\left(1-\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_{n-1}}\right)\tilde{U}_n^t\right).\tag{D.13}$$

The two other formulae are obtained by Weinstein–Aronstein duality.

Lemma 1.2 gives equation (7.7). □

Appendix E. Proof of theorem 7.3

The formula of [8] gives

$$W_n(x,y)=\det_n\left(\mathrm{Id}_n+\pi_{n-1}\frac{1}{x-Q}\frac{1}{y-P^t}\pi_{n-1}^t\right).\tag{E.1}$$

Using equation (5.10), it can be rewritten as

$$W_n(x,y) = \det_n \left(\mathrm{Id}_n + J_n(x,y)\pi_{n-1}\psi_\infty\phi^t_\infty\pi^t_{n-1} + \pi_{n-1}\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\pi^t_{n-1}\right) \quad \text{(E.2)}$$

$$W_{n+1}(x,y) = \det_{n+1} \left(\mathrm{Id}_{n+1} + J_n(x,y)\pi_n\psi_\infty\phi^t_\infty\pi^t_n + \pi_n\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\pi^t_n\right). \quad \text{(E.3)}$$

The difference is thus the same determinant with subtracting 1 in the n, n position:

$$W_{n+1}(x,y) - W_n(x,y) = \det_{n+1} \left(\Pi_{n-1} + J_n\pi_n\psi_\infty\phi^t_\infty\pi^t_n + \pi_n\frac{1}{x-Q}\Pi_{n-1}\frac{1}{y-P^t}\pi^t_n\right). \quad \text{(E.4)}$$

For any arbitrary non-vanishing α and $\tilde\alpha$, multiply the matrix inside the determinant (5.4) on the left by $\mathrm{Id}_{n+1} - \pi_n\frac{\psi_\infty\mathbf{e}^t_n}{\psi_n} + \alpha\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\psi_n}$ (whose determinant is $\frac{\alpha}{\psi_n}$) and on the right by $\mathrm{Id}_{n+1} - \frac{\mathbf{e}_n\phi^t_\infty}{\phi_n}\pi^t_n + \tilde\alpha\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\phi_n}$ (whose determinant is $\frac{\tilde\alpha}{\phi_n}$):

$$\left(\mathrm{Id}_{n+1} - \pi_n\frac{\psi_\infty\mathbf{e}^t_n}{\psi_n} + \alpha\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\psi_n}\right)\Pi_{n-1}\left(\mathrm{Id}_{n+1} - \frac{\mathbf{e}_n\phi^t_\infty}{\phi_n}\pi^t_n + \tilde\alpha\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\phi_n}\right) = \Pi_{n-1} = \mathrm{Id}_{n+1} - \mathbf{e}_n\,\mathbf{e}^t_n \quad \text{(E.5)}$$

$$\left(\mathrm{Id}_{n+1} - \pi_n\frac{\psi_\infty\mathbf{e}^t_n}{\psi_n} + \alpha\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\psi_n}\right)J_n\psi_\infty\phi^t_\infty\left(\mathrm{Id}_{n+1} - \frac{\mathbf{e}_n\phi^t_\infty}{\phi_n}\pi^t_n + \tilde\alpha\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\phi_n}\right) = J_n\alpha\tilde\alpha\,\mathbf{e}_n\,\mathbf{e}^t_n \quad \text{(E.6)}$$

$$\begin{aligned}
&\left(\mathrm{Id}_{n+1} - \pi_n\frac{\psi_\infty\mathbf{e}^t_n}{\psi_n} + \alpha\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\psi_n}\right)\pi_n\frac{1}{x-Q}\Pi_{n-1}\\
&\quad = \left(\Pi_{n-1}\left(1 - \frac{\psi_\infty\mathbf{e}^t_n}{\psi_n}\right) + \alpha\,\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\psi_n}\right)(R + \psi_\infty\hat\phi^t_\infty)\Pi_{n-1}\\
&\quad = \Pi_{n-1}\left(1 - \frac{\psi_\infty\mathbf{e}^t_n}{\psi_n}\right)\Pi_n R\Pi_{n-1} + \alpha\,\frac{\mathbf{e}_n\,\mathbf{e}^t_n}{\psi_n}R + \alpha\,\mathbf{e}_n\hat\phi^t_\infty\Pi_{n-1}\\
&\quad = \Pi_{n-1}\left(1 - \frac{\psi_\infty\mathbf{e}^t_n}{\psi_n}\right)\Pi_n R\Pi_{n-1} + \alpha\,\mathbf{e}_n T^t_n
\end{aligned} \quad \text{(E.7)}$$

where

$$T^t_n = \frac{\mathbf{e}^t_n R}{\psi_n} + \hat\phi^t_\infty\Pi_{n-1}, \qquad \tilde T_n = \frac{\tilde R^t\mathbf{e}_n}{\phi_n} + \Pi_{n-1}\hat\psi_\infty. \quad \text{(E.8)}$$

Finally, using equation (5.6) we have

$$\begin{aligned}
\frac{\alpha\tilde\alpha}{\psi_n\phi_n}(W_{n+1}(x,y) - W_n(x,y)) &= \det_{n+1}\big(\mathrm{Id}_{n+1} + (J_n\alpha\tilde\alpha - 1)\,\mathbf{e}_n\,\mathbf{e}^t_n\\
&\quad + \big((\Pi_{n-1}(x-Q)\Pi_{n-1})^{-1} + \alpha\;\mathbf{e}_n T^t_n\big)\big((\Pi_{n-1}(y-P^t)\Pi_{n-1})^{-1} + \tilde\alpha\tilde T_n\,\mathbf{e}^t_n\big)\big).
\end{aligned} \quad \text{(E.9)}$$

If we choose $\alpha\tilde\alpha = \frac{1}{J_n}$, we have

$$\begin{aligned}
\frac{1}{J_n\psi_n\phi_n}(W_{n+1}(x,y) - W_n(x,y)) &= \det_{n+1}\big(\mathrm{Id}_{n+1} + \big((\Pi_{n-1}(x-Q)\Pi_{n-1})^{-1} + \alpha\,\mathbf{e}_n T^t_n\big)\\
&\quad\times\big((\Pi_{n-1}(y-P^t)\Pi_{n-1})^{-1} + \tilde\alpha\tilde T_n\,\mathbf{e}^t_n\big)\big)\\
&= \det_n\big(\mathrm{Id}_n + \big((\Pi_{n-1}(y-P^t)\Pi_{n-1})^{-1} + \tilde\alpha\tilde T_n\,\mathbf{e}^t_n\big)\big((\Pi_{n-1}(x-Q)\Pi_{n-1})^{-1} + \alpha\,\mathbf{e}_n T^t_n\big)\big)\\
&= \det_n\left(\mathrm{Id}_n + (\Pi_{n-1}(y-P^t)\Pi_{n-1})^{-1}(\Pi_{n-1}(x-Q)\Pi_{n-1})^{-1} + \frac{1}{J_n}\tilde T_n T^t_n\right).
\end{aligned} \quad \text{(E.10)}$$

Then, we multiply both sides by equation (5.1), i.e. by $\det(\Pi_{n-1}(x-Q)\Pi_{n-1})$ and $\det(\Pi_{n-1}(y-P^t)\Pi_{n-1})$:

$$\frac{h_n\,\mathrm{e}^{V_1+V_2}}{J_n}(W_{n+1}(x,y)-W_n(x,y)) = \det_n\Big(\mathrm{Id}_n+\Pi_{n-1}(y-P^t)\Pi_{n-1}(x-Q)\Pi_{n-1} + \frac{1}{J_n}\Pi_{n-1}(y-P^t)\tilde{T}_n T_n^t(x-Q)\Pi_{n-1}\Big). \tag{E.11}$$

We have

$$\begin{aligned} T_n^t(x-Q)\Pi_{n-1} &= \frac{1}{\psi_n}\,\mathbf{e}_n^t R(x-Q)\Pi_{n-1} + \hat{\phi}_\infty^t\Pi_{n-1}(x-Q)\Pi_{n-1} \\ &= \frac{}{\psi_n}\,\mathbf{e}_n^t\left(1-\frac{\psi_\infty\,\mathbf{e}_0^t}{\psi_0}\right)\Pi_{n-1} + \hat{\phi}_\infty^t(x-Q) - \hat{\phi}_\infty^t\Pi^n(x-Q)\Pi_{n-1} \\ &= -\frac{\mathbf{e}_0^t}{\psi_0}+\frac{\mathbf{e}_0^t}{\psi_0} - \hat{\phi}_\infty^t\Pi^n(x-Q)\Pi_{n-1} \\ &= -\hat{\phi}_\infty^t\Pi^n(x-Q)\Pi_{n-1}. \end{aligned} \tag{E.12}$$

Also note that Heisenberg's relation equation (2.45) implies

$$\mathrm{Id}_n + \Pi_{n-1}(y-P^t)(x-Q)\Pi_{n-1} = \Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1}. \tag{E.13}$$

Thus,

$$\begin{aligned} \frac{h_n\,\mathrm{e}^{V_1+V_2}}{J_n}(W_{n+1}(x,y)-W_n(x,y)) &= \det_n\Big(\mathrm{Id}_n+\Pi_{n-1}(y-P^t)\Pi_{n-1}(x-Q)\Pi_{n-1} \\ &\quad + \frac{1}{J_n}\Pi_{n-1}(y-P^t)\Pi^n\hat{\psi}_\infty\hat{\phi}_\infty^t\Pi^n(x-Q)\Pi_{n-1}\Big) \\ &= \det_n\Big(\mathrm{Id}_n+\Pi_{n-1}(y-P^t)(x-Q)\Pi_{n-1} - \Pi_{n-1}(y-P^t)\Pi^n(x-Q)\Pi_{n-1} \\ &\quad + \frac{1}{J_n}\Pi_{n-1}(y-P^t)\Pi^n\hat{\psi}_\infty\hat{\phi}_\infty^t\Pi^n(x-Q)\Pi_{n-1}\Big) \\ &= \det_n\Big(\Pi_{n-1}(x-Q)(y-P^t)\Pi_{n-1} - \Pi_{n-1}(y-P^t) \\ &\quad \times\Pi^n\left(1-\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_n}\right)\Pi^n(x-Q)\Pi_{n-1}\Big). \end{aligned} \tag{E.14}$$

Now, use equations (5.4) and (5.8) that give

$$\begin{aligned} \frac{1}{J_nK_{n+1}}(W_{n+1}(x,y)-W_n(x,y)) &= \det_n\Big(\mathrm{Id}_n - \Pi_{n-1}(y-P^t)\Pi^n\left(1-\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_n}\right) \\ &\quad \times\Pi^n(x-Q)\Pi_{n-1}\tilde{R}^t\Pi_n\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)\Pi_nR\Pi_{n-1}\Big) \\ &= \det_{n+1}\Big(\mathrm{Id}_{n+1}-\Pi_n\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)\Pi_nR\Pi_{n-1}(y-P^t)\Pi^n\left(1-\frac{\hat{\psi}_\infty\hat{\phi}_\infty^t}{J_n}\right) \\ &\quad \times\Pi^n(x-Q)\Pi_{n-1}\tilde{R}^t\Pi_n\Big). \end{aligned} \tag{E.15}$$

Now, because of equation (2.47), we have

$$\begin{aligned}\Pi_n R\Pi_{n-1}(y-P^t)\Pi^n &= \Pi_n R V_1'(Q)\Pi^n \\ &= -\Pi_n \frac{V_1'(x)-V_1'(Q)}{x-Q}\Pi^n \\ &= -\mathcal{W}_n(x).\end{aligned} \tag{E.16}$$

Therefore,

$$W_{n+1}(x,y)-W_n(x,y) = J_n K_{n+1}\det_{n+1}\left(\mathrm{Id}_{n+1}-\Pi_n\left(1-\frac{\psi_\infty\phi_\infty^t}{K_{n+1}}\right)\mathcal{W}_n\left(1-\frac{\hat\psi_\infty\hat\phi_\infty^t}{J_n}\right)\tilde{\mathcal{W}}_n^t\right). \tag{E.17}$$

The second part of the formula is obtained by applying lemma 1.2.

Appendix F. Proof of theorem 8.1

We start from theorem 7.1:

$$\begin{aligned}W_n(x,y) = \gamma_n^2\, &\det(\mathrm{Id}_{n+1}-U_n(x,y)\tilde U_n(x,y)^t) \\ &\times\left(\left(J_{n+d_2}+\hat{\vec\phi}_n^t\frac{1}{1-\tilde U_n^t U_n}\hat{\vec\psi}_n\right)\left(K_{n-d_2}+\vec\phi_n^t\frac{1}{1-U_n\tilde U_n^t}\vec\psi_n\right)\right. \\ &\left.-\left(\hat{\vec\phi}_n^t\tilde U_n^t\frac{1}{1-U_n\tilde U_n^t}\vec\psi_n\right)\left(\vec\phi_n^t\frac{1}{1-U_n\tilde U_n^t}U_n\hat{\vec\psi}_n\right)\right).\end{aligned} \tag{F.1}$$

Let us compute the various terms:

•

$$\begin{aligned}\vec\phi_n^t\frac{1}{1-U_n\tilde U_n^t}\vec\psi_n &= \vec\phi_n^t\frac{1}{1-U_n\tilde U_n^t}\vec\psi_n \\ &= \vec\phi_n^t\tilde U_n^{t-1}A_n\Psi_n\frac{1}{(\tilde U_n^{t-1}-U_n)A_n\Psi_n}\vec\psi_n \\ &= -\phi_\infty^t B_{n-d_2}^t\Pi_{n-1}\Psi_\infty(x)\frac{1}{y-\Psi_n^{-1}\Psi_n'}f^{(0)} \\ &= -\phi_\infty^t(y)\Pi_{n-d_2}B_{n-d_2}^t\Pi_{n-1}\Psi_\infty(x)\frac{1}{y-\Psi_n^{-1}(x)\Psi_n'(x)}f^{(0)} \\ &= \mathrm{e}^{-V_2(y)}O(y^{n-d_2-1})\end{aligned} \tag{F.2}$$

where $f^{(0)}$ is the vector $(1,0,\ldots,0)^t$ of dimension d_2+1.

At large y, this expression behaves like $\mathrm{e}^{-V_2(y)}O(y^{n-d_2-1})$.

•

$$\begin{aligned}\hat\phi_\infty^t\Pi_{n+d_2-1}^{n-1}\frac{1}{1-\tilde U_n^t U_n}\Pi_{n+d_1-1}^{n-1}\hat\psi_\infty &= f^{(0)t}\hat\Phi_n^t\frac{1}{\tilde U_n^{t-1}-U_n}\tilde U_n^{t-1}\hat\Psi_n f^{(0)} \\ &= f^{(0)t}\hat\Phi_n^t A_n\Psi_n\frac{1}{(\tilde U_n^{t-1}-U_n)A_n\Psi_n}\tilde U_n^{t-1}\hat\Psi_n f^{(0)} \\ &= f^{(0)t}\frac{1}{y\Psi_n-\Psi_n'}\tilde U_n^{t-1}\hat\Psi_n f^{(0)} \\ &= f^{(0)t}\frac{1}{y-\Psi_n^{-1}\Psi_n'}\Psi_n^{-1}\tilde U_n^{t-1}\hat\Psi_n f^{(0)}\end{aligned}$$

$$= f^{(0)t}\frac{1}{y-\Psi_n^{-1}\Psi_n'}\hat{\Phi}_n^t A_n \tilde{U}_n^{t-1}\hat{\Psi}_n f^{(0)}$$
$$= -f^{(0)t}\frac{1}{y-\Psi_n^{-1}(x)\Psi_n'(x)}\hat{\Phi}_\infty^t(x)\Pi^n B_{n+d_2}^t\hat{\hat{\psi}}(y)$$
$$= \mathrm{e}^{V_2(y)}O(y^{-n-d_2-1}). \tag{F.3}$$

It behaves as $\mathrm{e}^{V_2(y)}O(y^{-n-d_2-1})$ at large y.

•

$$\hat{\phi}_\infty^t\tilde{U}_n^t\frac{1}{1-U_n\tilde{U}_n^t}\psi_\infty = f^{(0)t}\hat{\Phi}_n^t\frac{1}{\tilde{U}_n^{t-1}-U_n}\Psi_n f^{(0)}$$
$$= f^{(0)t}\hat{\Phi}_n^t A_n\Psi_n\frac{1}{\left(\tilde{U}_n^{t-1}-U_n\right)A_n\Psi_n}\Psi_n f^{(0)}$$
$$= f^{(0)t}\frac{1}{y\Psi_n-\Psi_n'}\Psi_n f^{(0)}$$
$$= f^{(0)t}\frac{1}{y-\Psi_n^{-1}\Psi_n'}f^{(0)}$$
$$= \left(\frac{1}{y-H_n(x,x)}\right)^{(00)}. \tag{F.4}$$

• Similarly, we have

$$\phi_\infty^t\frac{1}{1-U_n\tilde{U}_n^t}U_n\hat{\psi}_\infty = \left(\frac{1}{x-\tilde{H}_n(y,y)}\right)^{(00)}, \tag{F.5}$$

but unfortunately that formula is not sufficient to find the large y behaviour.

Instead, we prove the following lemma:

Lemma F.1.

$$(x+V_2'(y))\phi_\infty^t\frac{1}{1-U_n\tilde{U}_n^t}U_n\hat{\psi}_n = -1-\frac{n}{\tilde{t}y^{d_2+1}}+O(y^{-d_2-2}) \tag{F.6}$$

and symmetrically

$$(y+V_1'(x))\hat{\phi}_\infty^t\tilde{U}_n^t\frac{1}{1-U_n\tilde{U}_n^t}\psi_n = -1-\frac{n}{tx^{d_1+1}}+O(x^{-d_1-2}). \tag{F.7}$$

Proof. Compute the large y behaviour of the following matrix:

$$\Pi^{n-1}\left(\tilde{U}_n^t+(x+V_2')\hat{\psi}_n\phi_\infty^t\right)\Pi_n = \Pi^{n-1}\left(\tilde{R}^t(x-Q)+(x+V_2')\hat{\psi}_n\phi_\infty^t\right)\Pi_n$$
$$= \Pi^{n-1}\left(\left(\tilde{R}^t+\hat{\psi}_\infty\phi_\infty^t\right)(x-Q)+\hat{\psi}_n\phi_\infty^t\right.$$
$$\left.\times(V_2'(y)-V_2'(P^t))+\hat{\psi}_n\phi_\infty^t(V_2'(P^t)+Q)\right)\Pi_n$$
$$= \Pi^{n-1}\left(\frac{1}{y-P^t}(x-Q)+\hat{\psi}_n\phi_\infty^t(V_2'(P^t)+Q)\right)\Pi_n. \tag{F.8}$$

It is easy to see that this expression behaves like $O(y^{-1})$ at large y. The leading term in y is

$$\frac{1}{y}\Pi^{n-1}(x-Q)\Pi_n+\hat{\psi}_{n-1}\,\mathrm{e}^{-V_2(y)}\frac{\partial}{\partial y}(\mathrm{e}^{V_2(y)}\phi_n(y))\,\mathbf{e}_{n-1}\,\mathbf{e}_n^t+O(y^{-2})$$
$$= \frac{1}{y}\Pi^{n-1}(x-Q)\Pi_n+\sqrt{h_{n-1}}y^{-n}\frac{n}{\sqrt{h_n}}y^{n-1}\mathbf{e}_{n-1}\mathbf{e}_n^t+O(y^{-2})$$
$$= \frac{1}{y}\left(\Pi^{n-1}(x-Q)\Pi_n+\frac{n}{\gamma_n}\,\mathbf{e}_{n-1}\,\mathbf{e}_n^t\right)+O(y^{-2}) \tag{F.9}$$

and thus, using $U_n = -\frac{y}{\gamma_n}\,\mathbf{e}_n\,\mathbf{e}^t_{n-1} + O(1)$, we have

$$U_n\big(\tilde U_n^t + (x+V_2')\hat\psi_n\phi_\infty^t\big)\Pi_n = -\frac{1}{\gamma_n}\,\mathbf{e}_n\,\mathbf{e}^t_{n-1}\left((x-Q)\Pi_n + \frac{n}{\gamma_n}\,\mathbf{e}_{n-1}\,\mathbf{e}^t_n\right) + O(y^{-1})$$
$$= -\frac{1}{\gamma_n}\,\mathbf{e}_n\left(\mathbf{e}^t_{n-1}(x-Q)\Pi_n + \frac{n}{\gamma_n}\,\mathbf{e}^t_n\right) + O(y^{-1}). \tag{F.10}$$

and the following determinant is

$$\det\big(1 - U_n\big(\tilde U_n^t + (x+V_2')\hat\psi_n\phi_\infty^t\big)\Pi_n\big)$$
$$= \det\left(1 + \frac{1}{\gamma_n}\,\mathbf{e}_n\left(\mathbf{e}^t_{n-1}(x-Q)\Pi_n + \frac{n}{\gamma_n}\mathbf{e}^t_n\right) + O(y^{-1})\right)$$
$$= 1 + \frac{1}{\gamma_n}\left(\mathbf{e}^t_{n-1}(x-Q)\,\mathbf{e}_n + \frac{n}{\gamma_n}\right) + O(y^{-1})$$
$$= 1 + \left(\frac{n}{\gamma_n^2} - 1\right) + O(y^{-1})$$
$$= \frac{n}{\gamma_n^2} + O(y^{-1}). \tag{F.11}$$

It follows

$$\frac{n}{\gamma_n^2} + O(y^{-1}) = \det\big(1 - U_n\big(\tilde U_n^t + (x+V_2')\hat\psi_n\phi_\infty^t\big)\Pi_n\big)$$
$$= \det\big(1 - U_n\big(\tilde U_n^t + (x+V_2')U_n\hat\psi_n\phi_\infty^t\big)\Pi_n\big)$$
$$= \det\big(1 - U_n\tilde U_n^t\big)\left(\det\left(1 + (x+V_2')\frac{1}{1-U_n\tilde U_n^t}U_n\hat\psi_n\phi_\infty^t\right)\Pi_n\right)$$
$$= \det\big(1 - U_n\tilde U_n^t\big)\left(1 + (x+V_2')\phi_\infty^t\frac{1}{1-U_n\tilde U_n^t}U_n\hat\psi_n\right) \tag{F.12}$$

that implies

$$(x+V_2')\phi_\infty^t\frac{1}{1-U_n\tilde U_n^t}U_n\hat\psi_n = -1 + \frac{n}{\gamma_n^2\det\big(1-U_n\tilde U_n^t\big)} + O(y^{-d_2-2})$$
$$= -1 - \frac{n}{\mathcal{E}_n(x,y)} + O(y^{-d_2-2})$$
$$= -1 - \frac{n}{\tilde t\, y^{d_2+1}} + O(y^{-d_2-2}). \tag{F.13}$$

Similarly, we have

$$(y+V_1'(x))\hat\phi_\infty^t\tilde U_n^t\frac{1}{1-U_n\tilde U_n^t}\psi_n = -1 - \frac{n}{t x^{d_1+1}} + O(x^{-d_2-2}). \tag{F.14}$$

□

F.1. U-conjecture, lemma 8.1

Using lemma F.1, we have

$$(x+V_2'(y))\det\big(1-U_n\tilde U_n^t\big)\left(\hat\phi_\infty^t\tilde U_n^t\frac{1}{1-U_n\tilde U_n^t}\psi_n\right)\left(\phi_\infty^t\frac{1}{1-U_n\tilde U_n^t}U_n\hat\psi_n\right)$$
$$= -\det\big(1-U_n\tilde U_n^t\big)\left(\hat\phi_\infty^t\tilde U_n^t\frac{1}{1-U_n\tilde U_n^t}\psi_n\right) - \frac{n}{\gamma_n^2}\left(\hat\phi_\infty^t\tilde U_n^t\frac{1}{1-U_n\tilde U_n^t}\psi_n\right) + O(y^{-2})$$

$$= -\det\left(1-U_n\tilde U_n^t\right)\left(\frac{1}{y-H_n(x,x)}\right)^{(00)} - \frac{n}{\gamma_n^2}\left(\frac{1}{y-H_n(x,x)}\right)^{(00)} + O(y^{-2})$$
$$= \frac{\tilde t}{\gamma_n^2}\det(y-H_n(x,x))\left(\frac{1}{y-H_n(x,x)}\right)^{(00)} - \frac{n}{\gamma_n^2}\left(\frac{1}{y-H_n(x,x)}\right)^{(00)} + O(y^{-2})$$
$$= \frac{\tilde t}{\gamma_n^2}\det(y-H_n(x,x))\left(\frac{1}{y-H_n(x,x)}\right)^{(00)} + O(y^{-1}). \tag{F.15}$$

The first term is a polynomial in y and this proves theorem 8.3.

Appendix G. Proof of theorem 8.3

Starting from lemma F.1, we have

$$\gamma_n^2\det\left(1-U_n\tilde U_n^t\right)(y+V_1'(x))\left(\hat\phi_\infty^t\tilde U_n^t\frac{1}{1-U_n\tilde U_n^t}\psi_n\right) = \mathcal{E}_n(x,y)\left(1+\frac{n}{tx^{d_1+1}}\right)+O(x^{-1})$$
$$= \mathcal{E}_n(x,y)+n+O(x^{-1}) \tag{G.1}$$

from which theorem 8.3 follows.

Appendix H. Proof of theorem 8.2

Let $r_n(x)$ be the matrix:

$$r_n(x) = \Pi_{n+1}^{n-d_2+1}F_n(x) = \Pi_{n+1}^{n-d_2+1} + \frac{1}{\gamma_{n+1}}\mathbf{e}_{n+1}\,\mathbf{e}_n^t(x-Q). \tag{H.1}$$

By definition, it is such that

$$\Psi_{n+1}(x) = r_n(x)\Psi_n(x). \tag{H.2}$$

Thus,

$$\mathcal{D}_{n+1}(x) = \Psi_{n+1}'\Psi_{n+1}^{-1} = (r_n\Psi_n)'(r_n\Psi_n)^{-1}$$
$$= r_n'r_n^{-1} + r_n\mathcal{D}_nr_n^{-1}$$
$$= r_n\left(r_n^{-1}r_n' + \mathcal{D}_n\right)r_n^{-1}. \tag{H.3}$$

Therefore,

$$\mathcal{E}_{n+1} = \tilde t\det\left(y-\mathcal{D}_n-r_n^{-1}r_n'\right). \tag{H.4}$$

Now, note that $r_n^{-1}r_n' = -\frac{1}{Q_{n,n-d_2}}\mathbf{e}_{n-d_2}\,\mathbf{e}_n^t$, thus

$$\mathcal{E}_{n+1}-\mathcal{E}_n = (-1)^{d_2+1}\frac{\tilde t}{Q_{n,n-d_2}}\det(\mathcal{C}_n) \tag{H.5}$$

where we have defined

$$\mathcal{C}_n(x,y) = \Pi_n^{n-d_2+1}(y+\mathcal{D}_n(x))\Pi_{n-1}^{n-d_2}. \tag{H.6}$$

We have

$$\mathcal{C}_n = \Pi_n^{n-d_2+1}(y-P^t)\Pi_{n-1}^{n-d_2} + \Pi_n^{n-d_2+1}\frac{V_1'(x)-V_1'(Q)}{x-Q}(1-\Pi_{n-1})(x-Q)\Pi_{n-1}^{n-d_2}$$
$$= \Pi_n^{n-d_2+1}(y-P^t)\Pi_{n-1}^{n-d_2} + \Pi_n^{n-d_2+1}\mathcal{W}_n(x-Q)\Pi_{n-1}^{n-d_2} \tag{H.7}$$

and

$$\mathcal{C}_n\Pi_{n-1}^{n-d_2}\tilde R^t\Pi_n^{n-d_2+1} = \Pi_n^{n-d_2+1}\left(1-\mathcal{W}_n\tilde{\mathcal{W}}_n^t\right)\Pi_n^{n-d_2+1}. \tag{H.8}$$

That implies

$$\det(\mathcal{C}_n)\Pi_{n-1}^{n-d_2}\tilde{R}^t\Pi_n^{n-d_2+1} = (-1)^{d_2}\gamma_n\cdots\gamma_{n-d_2+1}\det\left(1-\mathcal{W}_n\tilde{\mathcal{W}}_n^t\right)$$
$$= (-1)^{d_2+1}\frac{Q_{n,n-d_2}}{\tilde{t}}\det\left(1-\mathcal{W}_n\tilde{\mathcal{W}}_n^t\right) \tag{H.9}$$

which proves the formula.

Appendix I. A useful formula for determinants with rank 2 matrices

Lemma I.2. *If M is an invertible matrix and a, b, c, d are arbitrary vectors, we have*

$$\det(M+ab^t+cd^t) = \det M\left(\left(1+b^t\frac{1}{M}a\right)\left(1+d^t\frac{1}{M}c\right)-b^t\frac{1}{M}cd^t\frac{1}{M}a\right). \tag{I.1}$$

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